Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2023 (S62)

Solutions to Assignment #1
Simulations

Something that comes up in a lot of practical applications of probability is simulating some desired random process by the one you actually have access to. The classic problem along these lines is wanting to have a fair coin, but only having a biased coin with even the bias not being known†. One can work around this by using the following process to simulate a fair coin:

1. Toss the biased coin twice.
2. If the outcome of the two coin tosses if step 1 is HH or TT, repeat step 1. Otherwise, return H for the simulated coin if the outcome of step 1 was HT, and return T for the simulated coin if the outcome of step 1 was TH.

Why does this process work? As long as the outcomes HH and TT of the biased coin have a combined probability less than 1, the probability that you will get HH and/or TT forever is zero, so the process will eventually terminate and return some outcome for the simulated fair coin. The two simulated outcomes are equally likely because however the coin being used is biased, the outcomes HT and TH are equally likely. (Why?)

In this assignment you will be asked to use some given random process or tool to simulate another one.

1. Explain how to use a fair standard six-sided die to simulate a fair coin. *[1]*

Solution. Here is a process or experiment that uses a fair standard six-sided die to simulate a fair coin:

1. Roll the fair die once.
2. If the die came up with the face numbered 1, 2, or 3, return H for the simulated coin, and if the die came up with the face numbered 4, 5, or 6, return T for the simulated coin.

This works because the die is fair, so probability of each face of the die coming up is $\frac{1}{6}$. This means that the probability of returning H is $P(H) = P(1)+P(2)+P(3) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{3}{6} = \frac{1}{2}$, and the probability of returning T is $P(T) = P(5)+P(5)+P(6) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{3}{6} = \frac{1}{2}$, too. □

2. Explain how to use a fair coin to simulate a fair standard six-sided die. *[3]*

Solution. Here is a process or experiment that uses a fair coin to simulate a fair standard six-sided die:

1. Toss the fair coin three times.
2. If the three tosses came up TTH or TTT, repeat step 1. Otherwise,
   if the tosses came up HHH, return 1 for the simulated die;
   if the tosses came up HHT, return 2 for the simulated die;

† A real-world version of this problem, faced by most operating systems, is using the output of some hardware random number generator, which is likely to have some bias, to generate unbiased random numbers.

if the tosses came up HTH, return 3 for the simulated die;
if the tosses came up HTT, return 4 for the simulated die;
if the tosses came up THH, return 5 for the simulated die; and
if the tosses came up THT, return 6 for the simulated die;

Since the coin is fair, all eight possible sequences of three tosses are equally likely, so each one has a probability of $\frac{1}{8} = 0.125$ of occuring. One consequence is that the combined probability of the sequences TTH and TTT is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4} < 1$, from which it follows that the probability of having to repeat step 1 forever is 0. Another consequence is that the other six sequences are all equally likely to turn up, so the faces of the simulated die are all equally likely to come up, *i.e.* the simulated die is fair. $\square$

3. Explain how to use a fair coin to simulated a biased coin which has a probability, on any given toss, of $\frac{1}{3}$ of the head coming, of $\frac{1}{2}$ of the tail coming up, and $\frac{1}{6}$ of landing (and staying!) on its edge. *[3]*

SOLUTION. We'll do this using an intermediate step: using a fair standard six-sided die to simulate the biased coin. Here is that process:

1. Roll the fair die once.
2. If the die came up with the face numbered 1 or 2, return H for the simulated biased coin;
   if the die came up with the face numbered 3, 4, or 5, return T for the simulated biased coin; and
   if the die came up with the face numbered 6, return E (*i.e.* edge) for the simulated biased coin.

Since the die is fair, each face has an equal likelihood of $\frac{1}{6}$ of coming up. It follows that the probabilities for the simulated biased coin are $P(H) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$, $P(T) = P(3) + P(4) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$, and $P(E) = P(6) = \frac{1}{6}$, as desired.

How do we finish the job? From question **2** we know how to simulate a fair die using a fair coin, so we use a fair coin to simulate the fair die in the process above that simulates the biased coin. $\square$

4. Explain how to use a fair ten-sided die, with its faces numbered 0 through 9, to simulate a biased coin that has a probability of $\frac{1}{\pi}$ of coming up heads and a probability of $1 - \frac{1}{\pi}$ of coming up tails. *[3]*

NOTE. In question **4**, please keep in mind that, like $\sqrt{2}$, $\frac{1}{\pi}$ is an irrational number, *i.e.* it not equal to $\frac{a}{b}$ for any integers a and b.

SOLUTION. As noted above, $\frac{1}{\pi} = 0.318309886184\ldots$ is an irrational number, so we can't express it as a ratio of integers, which means that the sort of tricks used to answer questions **2** and **3** above, just won't work. We will instead use the decimal expansion $\frac{1}{\pi}$ as the basis for our process. Since $\frac{1}{\pi}$ is irrational, this expansion runs on forever and never settles into a pattern that repeats forever past some point. However, since we can, in principle, compute the decimal expansion of $\pi = 3.14159265359\ldots$ to any finite extent we like, we can do the same for $\frac{1}{\pi}$. In he process described below, at any given stage we will need only a finite part of the decimal extension of $\frac{1}{\pi}$.

Here is the process, which will also include a counter n that will use to keep track of which decimal place we're dealing with:

1. Set $n = 1$.
2. Roll the fair ten-sided die.
3. If the number on the face that comes up is
   - less than the nth digit of $\frac{1}{\pi}$, return H for the simulated coin;
   - greater than the nth digit of $\frac{1}{\pi}$, return T for the simulated coin; and
   - equal to the nth digit of $\frac{1}{\pi}$, increase n by 1 and do steps 2 and 3 again (with the new value of n).

Why does this work?

First, observe that the probability that the process will run on forever is 0. This follows from the fact that matching the first digit of $\frac{1}{\pi}$ when rolling the die has probability $\frac{1}{10} = 0.1$, matching the first two digits by rolling the die twice has probability $\left(\frac{1}{10}\right)^2 = \frac{1}{100} = 0.01$, matching the first three digits by rolling the die thrice has probability $\left(\frac{1}{10}\right)^3 = \frac{1}{1000} = 0.001$, and so on.

Second, consider the thought experiment of rolling the die infinitely many times. Any infinite sequence of decimal digits is as likely to turn up as any other because the die is fair. Sliding a subtlety or two under the rug, if we think of real numbers between 0 and 1 as being given by their decimal expansions after the decimal point, this amounts to generating the real numbers in the interval $[0, 1]$ with equal likelihood. The likelihood of getting a number between 0 and $\frac{1}{\pi}$ *versus* getting a number between $\frac{1}{\pi}$ and 1 depends on the lengths of those intervals, which are $\frac{1}{\pi}$ and $1 - \frac{1}{\pi}$, respectively. (We can't usefully try to directly count the infinitely many real numbers in most contexts, but we *can* measure the lengths of intervals of intervals of real numbers.) It follows that the probability of the thought experiment generating a real number between 0 and $\frac{1}{\pi}$ is $\dfrac{\frac{1}{\pi}}{\frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right)} = \dfrac{1}{\pi}$, and of generating a a real number between $\frac{1}{\pi}$ and $1 - \frac{1}{\pi}$ is $\dfrac{1 - \frac{1}{\pi}}{\frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right)} = 1 - \dfrac{1}{\pi}$.

Third, consider what the process given above does in relation to the thought experiment. It generates just enough of the random real number in the thought experiment to determine whether that number falls between 0 and $\frac{1}{\pi}$ or between $\frac{1}{\pi}$ and 1.

It follows that process given above uses a fair ten-sided die, with its faces numbered 0 through 9, to simulate a biased coin that has a probability of $\frac{1}{\pi}$ of coming up heads and a probability of $1 - \frac{1}{\pi}$ of coming up tails, as desired. $\square$