Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2020 (S62)

Solutions to Assignment *e*

[The make-up for Assignment #2]

Alea iacta est!*

This optional assignment, should you choose to do it and do better than you did on Assignment #2, would replace your Assignment #2 in the marking scheme.

An unscrupulous gambler has modified a standard six-sided die so that the faces numbered 1 and 6 are each half as likely to come up as each of the other four faces when the die is rolled. Your task is to design experiments that let you use this die in place of

1. A fair coin. *[1]*

Solution. First, let's work out what the probabilities for each face on the die to come up when it's rolled really are. If $P(2) = P(3) = P(4) = P(5) = x$, then $P(1) = P(6) = \frac{x}{2}$. Since one of the six faces of the crooked die must come when it is rolled, we have

$$1 = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = \frac{x}{2} + x + x + x + x + \frac{x}{2} = 5x\,,$$

so $x = \frac{1}{5}$. Thus $P(2) = P(3) = P(4) = P(5) = \frac{1}{5} = 0.2$ and $P(1) = P(6) = \frac{1}{10} = 0.1$.

Our experiment to simulate the fair coin using the crooked die is simply to roll the die once. If it comes up with 1, 2, or 3, we record a head for the simulated coin; if it comes up 4, 5, or 6, we record a tail for the simulated coin. This works because

$$\begin{aligned} P(H) &= P(1) + P(2) + P(3) = \frac{1}{10} + \frac{1}{5} + \frac{1}{5} = \frac{1}{2} \\ \text{and } P(T) &= P(4) + P(5) + P(6) = \frac{1}{5} + \frac{1}{5} + \frac{1}{10} = \frac{1}{2}\,, \end{aligned}$$

so the coin being simulated by this experiment is fair. □

2. A biased coin with $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. *[2]*

Solution. Here is an experiment to simulate a biased coin with $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$ using the crooked die:

> Roll the crooked die. If it comes up 2, 3, or 4, record a head for simulated biased coin; if it comes up 5 or 6, record a tail for the simulated biased coin; and if it comes up 1, repeat the experiment.

* "The die is cast." Supposedly spoken by Gaius Julius Caesar on crossing the Rubicon river into Italy, setting off the civil wars that ended the Roman Republic and eventually led to the establishment of the Roman Empire.

Why does this work? The odds of a head *vs.* a tail are $P(2) + P(3) + P(4) = \frac{6}{10}$ to $P(5) + P(6) = \frac{3}{10}$, which is to say $2 : 1$, as desired for the biased coin. Note that the probability that the experiment will have to be repeated forever is the probability that 1 will come up infinitely many times in a row, namely $\lim_{n\to\infty} \left(\frac{1}{10}\right)^n = 0$. □

3. A fair standard die. *[3]*

Solution. We know from **1** that we can use the crooked die to simulate a fair coin. If we can show how to use a fair coin to simulate a fair die, we'll be done. (Execute the latter experiment using the former one as a "subroutine" …) Here is an experiment to simulate a fair die using a fair coin:

> Toss the fair coin three times. Assign the outcome HHH to the roll of a 1 of the die, HHT to the roll of a 2 of the die, HTH to the roll of a 3 of the die, THH to the roll of 4 of the die, HTT to the roll of a 5 of the die, and THT to the roll of a 6 of the die. If the outcome is TTH or TTT, repeat the experiment.

Since the coin is fair, each of the eight outcomes of tossing the coin three times is equally likely. This means that each face of the die is a likely to come up as any other, as required of a fair die. Note that the probability that the experiment will have to be repeated forever is the probability that TTH or TTT will come up infinitely many times in a row, namely $\lim_{n\to\infty} \left(\frac{1}{8} + \frac{1}{8}\right)^n = \lim_{n\to\infty} \left(\frac{1}{4}\right)^n = 0$. □

and, finally,

4. Design an experiment to use the modified die to pick an integer $n \geq 0$ such that the probability of choosing n is $\frac{1}{2^{n+1}}$. *[4]*

Solution. We know from **1** that we can use the crooked die to simulate a fair coin. If we can show how to use a fair coin to pick an integer $n \geq 0$ such that the probability of choosing n is $\frac{1}{2^{n+1}}$, we'll be done. Here is the experiment using a fair coin to pick the $n \geq 0$:

> Toss the fair coin until it comes up heads. If it came up heads on the kth toss, return $n = k - 1$.

The probability of selecting $n \geq 0$ by this method is the probability that a head came up for the first time on the $k = n + 1$st toss, *i.e.* the probability of the outcome T^nH. Since the coin is fair, $P\left(T^nH\right) = \left(\frac{1}{2}\right)^n \frac{1}{2} = \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2^{n+1}}$, as required.

Note, once again, that the probability of the experiment never ending is the proability of getting an infinite sequence of tails, namely $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$. ■

In each question, make sure to fully explain why your experiment meets the given requirements.