Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2020 (S62)

Solutions to Assignment #4
(Un)expected Values?
Due on Friday, 17 July.

Consider the probability density function $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$.

1. Verify that $f(x)$ is a valid probability density function. *[4]*

Solution. First, since $1 + x^2 \geq 1 > 0$ for all x and $\frac{1}{\pi} > 0$, $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} > 0$ for all x.

Second, since rational functions like $f(x)$ are continuous everywhere they are defined, and $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ is defined everywhere (because $1 + x^2 \geq 1 > 0$ for all x, the denominator is never 0), $f(x)$ is continuous and thus integrable for all x.

Third, we check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For once, we will do the improper integral carefully. Look up the antiderivative if you don't already know it. If you took MATH 1005H you are likely not to have seen the inverse to $\tan(x)$, namely $\arctan(x)$, which is the antiderivative of $\frac{1}{1+x^2}$.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{1}{1+x^2}\,dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \\
&= \frac{1}{\pi} \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to\infty} \frac{1}{\pi} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \frac{1}{\pi} \int_{0}^{b} \frac{1}{1+x^2}\,dx \\
&= \frac{1}{\pi} \lim_{a\to\infty} \arctan(x)\Big|_a^0 + \frac{1}{\pi} \lim_{b\to\infty} \arctan(x)\Big|_0^b \\
&= \frac{1}{\pi} \lim_{a\to\infty} \left(\arctan(0) - \arctan(a)\right) + \frac{1}{\pi} \lim_{b\to\infty} \left(\arctan(b) - \arctan(0)\right) \\
&= \frac{1}{\pi}\left(0 - \left(-\frac{\pi}{2}\right)\right) + \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

Since it satisfies all three conditions necessary to be a valid probability density, $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ is one. □

2. Suppose the continuous random variable X has $f(x)$ as its density function. Show that $E(X)$ is undefined. *[2]*

SOLUTION. We try to compute the expected value of X and see what happens. Once again, we'll do the improper integral fairly carefully.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{x}{1+x^2}\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx \\
&= \frac{1}{\pi} \int_{-\infty}^{0} \frac{x}{1+x^2}\, dx + \frac{1}{\pi} \int_{0}^{\infty} \frac{x}{1+x^2}\, dx \\
&= \lim_{a \to -\infty} \frac{1}{\pi} \int_{a}^{0} \frac{x}{1+x^2}\, dx + \lim_{b \to \infty} \frac{1}{\pi} \int_{0}^{b} \frac{x}{1+x^2}\, dx
\end{aligned}
$$

Substitute $u = 1 + x^2$, so $du = 2x\, dx$ and $x\, dx = \frac{1}{2}\, du$, and change the limits of integration as we go along:

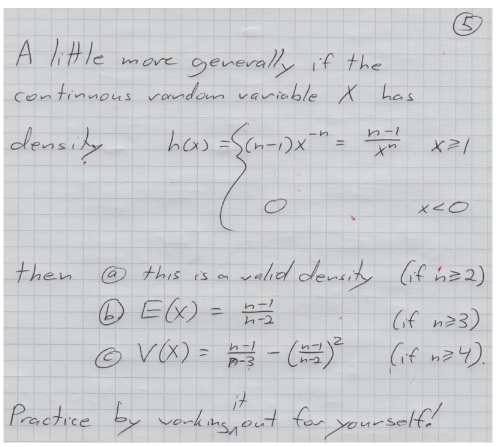
x	a	0	b
u	$1+a^2$	1	$1+b^2$

$$
\begin{aligned}
&= \lim_{a \to -\infty} \frac{1}{\pi} \int_{1+a^2}^{1} \frac{1}{u} \cdot \frac{1}{2}\, du + \lim_{b \to \infty} \frac{1}{\pi} \int_{1}^{1+b^2} \frac{1}{u} \cdot \frac{1}{2}\, du \\
&= \frac{1}{2\pi} \lim_{a \to -\infty} \ln(u)\big|_{1+a^2}^{1} + \frac{1}{2\pi} \lim_{b \to \infty} \ln(u)\big|_{1}^{1+b^2} \\
&= \frac{1}{2\pi} \lim_{a \to -\infty} \left(\ln(1) - \ln\left(1+a^2\right)\right) + \frac{1}{2\pi} \lim_{b \to \infty} \left(\ln\left(1+b^2\right) - \ln(1)\right) \\
&= -\frac{1}{2\pi} \lim_{a \to -\infty} \ln\left(1+a^2\right) + \frac{1}{2\pi} \lim_{b \to \infty} \ln\left(1+b^2\right)
\end{aligned}
$$

At this point, things break down: as $1+x^2 \to \infty$, and hence $\ln\left(1+x^2\right) \to \infty$, as $x \to \pm\infty$, neither limit needed to compute the improper integral equals a real number. Since both need to equal a real number in order for the improper integral we started with to be defined, it follows that $E(X)$ is undefined. □

3. Find an example of a probability density function $g(x)$ such that if a continuous random variable X has $g(x)$ as its density function, then $E(X)$ is defined but $V(X)$ is not. *[4]*

SOLUTION. We take a little hint from the end of the lecture *Example: A pretty easy continuous density function or two*:


(5)

A little more generally if the continuous random variable X has density $h(x) = \begin{cases} (n-1)x^{-n} = \frac{n-1}{x^n} & x \geq 1 \\ 0 & x < 0 \end{cases}$

then (a) this is a valid density (if $n \geq 2$)

(b) $E(X) = \frac{n-1}{n-2}$ (if $n \geq 3$)

(c) $V(X) = \frac{n-1}{n-3} - \left(\frac{n-1}{n-2}\right)^2$ (if $n \geq 4$).

Practice by working it out for yourself!

Since $E(X)$ seems to be defined for this family of densities when $n \geq 3$ and $V(X)$ is only defined when $n \geq 4$, lets try the case $n = 3$, so we'll let $g(x) = \begin{cases} 2x^{-3} & x \geq 1 \\ 0 & x < 0 \end{cases}$. We do need to check that this is indeed a valid density:

First, since $0 \geq 0$ and $2x^{-3} > 0$ when $x \geq 1$, it follows that $g(x) \geq 0$ for all x.

Second, since $g(x)$ is defined and continuous everywhere except at $x = 1$, where it has a jump discontinuity, $g(x)$ is integrable everywhere.

Third,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} 2x^{-3}\,dx = 0 + \lim_{a\to\infty} \int_{1}^{a} 2x^{-3}\,dx \\
&= \lim_{a\to\infty} 2 \cdot \left.\frac{x^{-3+1}}{-3+1}\right|_1^a = \lim_{a\to\infty} 2 \cdot \left.\frac{x^{-2}}{-2}\right|_1^a = \lim_{a\to\infty} -x^{-2}\Big|_1^a \\
&= \lim_{a\to\infty} \left(\left(-a^{-2}\right) - \left(-1^{-2}\right)\right) = \lim_{a\to\infty} \left(-\frac{1}{a^2} + 1\right) = -0 + 1 = 1\,,
\end{aligned}$$

since $\frac{1}{a^2} \to 0$ as $a \to \infty$.

Since it satisfies all the necessary conditions, $g(x)$ is a valid density function.

Next, let us compute $E(X)$ for a random variable X having $g(x)$ as its density:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xg(x)\,dx = \int_{-\infty}^{1} x \cdot 0\,dx + \int_{1}^{\infty} x \cdot 2x^{-3}\,dx = 0 + \lim_{a\to\infty} \int_{1}^{a} 2x^{-2}\,dx \\
&= \lim_{a\to\infty} 2 \cdot \left.\frac{x^{-2+1}}{-2+1}\right|_1^a = \lim_{a\to\infty} 2 \cdot \left.\frac{x^{-1}}{-1}\right|_1^a = \lim_{a\to\infty} -2x^{-1}\Big|_1^a = \lim_{a\to\infty} \left.\frac{-2}{x}\right|_1^a \\
&= \lim_{a\to\infty} \left(\frac{-2}{a} - \frac{-2}{1}\right) = \lim_{a\to\infty} \left(\frac{-2}{a} + 2\right) = -0 + 2 = 2
\end{aligned}$$

Since we got a real number out of this computation, $E(X)$ is defined.

Finally, let us try to compute $V(X) = E\left(X^2\right) - [E(X)]^2$. We have $E(X) = 2$ from the above, so it remains to compute $E\left(X^2\right)$.

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 g(x)\,dx = \int_{-\infty}^{1} x^2 \cdot 0\,dx + \int_{1}^{\infty} x^2 \cdot 2x^{-3}\,dx = 0 + \lim_{a\to\infty} \int_{1}^{a} 2x^{-1}\,dx \\
&= \lim_{a\to\infty} \int_{1}^{a} \frac{2}{x}\,dx = \lim_{a\to\infty} 2\ln(x)\big|_1^a = \lim_{a\to\infty} \left(2\ln(a) - 2\ln(1)\right) = \lim_{a\to\infty} \left(2\ln(a) - 2 \cdot 0\right) \\
&= \lim_{a\to\infty} 2\ln(a) = \infty\,,
\end{aligned}$$

since $\ln(a) \to \infty$ as $a \to \infty$. Since we did not get a real number out of the improper integral, $E\left(X^2\right)$, and consequently also $V(X)$, is undefined. ■