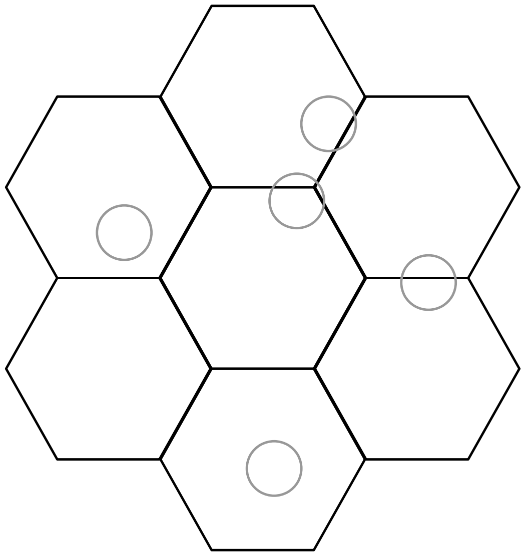
Mathematics 1550H – Probability I: Introduction to Probability

Trent University, Summer 2020 (S62)

Solution to Assignment #3
Hexenkreis

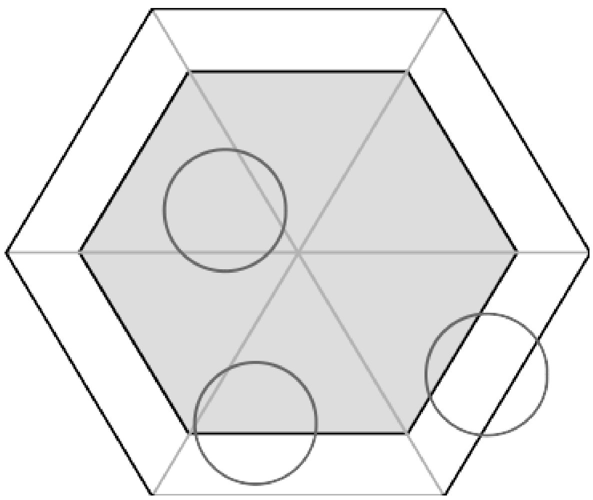


We are given an infinite grid of regular hexagons with side length 2, of which a small piece is given above. Coins of diameter 1 are randomly tossed onto this grid.

1. What is the probability that a given coin randomly tossed onto the grid will not end up touching the side of one or more hexagons? *[10]*

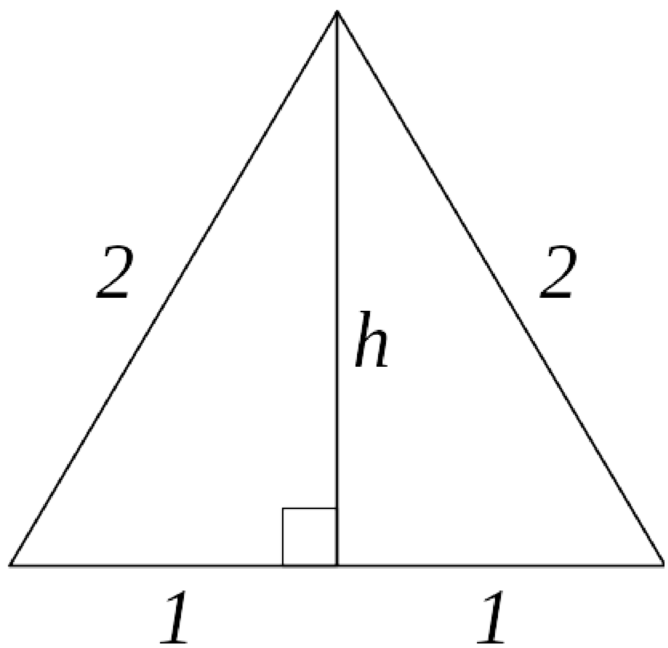
Hint: This problem is similar in concept to Exercise 15 in Section 2.2 of the textbook. You may need to compute an area or two, but you won't need calculus. The area of a regular hexagon, should you need it, is something you can look up (in which case give a reference) or work out using basic geometry (in which case you should give at least a sketch of how you worked it out).

Solution. We only need to consider the hexagon the centre of the coin ends up in:



The coin will not touch the sides of the hexagon the centre of the coin lands in if the centre of the coin is at least one radius ($= \frac{1}{2}$) away from the sides of the hexagon. This gives a region that is a smaller hexagon, with the same centre as the larger hexagon, whose sides are parallel to the sides of the larger hexagon. The probability that the randomly tossed coin will not touch a side of the larger hexagon is the ratio of the area of the smaller hexagon to the area of the larger hexagon.

To compute the ratio of areas we need to know the dimensions of the smaller hexagon. Observe that a regular hexagon can be decomposed into six equilateral triangles whose side lengths are equal to the side length of the hexagon. Consider one such triangle for the larger hexagon, whose side length is 2, as in the diagram below.



The height h of this equilateral triangle is the length of the line joining the tip of the triangle to the midpoint of the opposite side. This line is a short side of either one of the two right triangles the line divides the equilateral triangle into. Since the other short side of one of these triangles has length 1 and the hypotenuse has length 2, we have $h^2+1^2 = 2^2$ by the Pythagorean Theorem, so $h = \sqrt{4-1} = \sqrt{3}$.

Note that $h = \sqrt{3}$ is the (closest) distance from the centre of the larger hexagon to its sides. The smaller hexagon has sides that are $\frac{1}{2}$ closer to the common center of the two hexagons; that is, the (closest) distance from the common centre of the two hexagons to the sides of the smaller hexagon is $\sqrt{3} - \frac{1}{2}$. Since the two hexagons are the same except for scale, the areas of the two hexagons, smaller to larger, are in the same proportion as the square of the proportion of their linear dimensions, *i.e.* in the ratio $\left(\dfrac{\sqrt{3}-\frac{1}{2}}{\sqrt{3}}\right)^2 = \dfrac{3-\sqrt{3}+\frac{1}{4}}{3} = 1 - \dfrac{1}{\sqrt{3}} + \dfrac{1}{12} = \dfrac{13}{12} - \dfrac{1}{\sqrt{3}}$.

Since the ratio of the areas is the probability we want, it follows that the probability that a coin of diameter 1 randomly tossed onto an infinite regular hexagonal grid with side length 2 will not end up touching one of the sides is $\frac{13}{12} - \frac{1}{\sqrt{3}} \approx 0.50598$. ■