Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2020 (S62)
Solutions to Quiz #6
Tuesday, 28 July

We roll a fair standard die 100 times, each roll being independent of all the others. For $1 \le i \le 100$, let the random variable X_i give the number on the face that comes up on the ith die roll. Let $X = \sum_{n=1}^{100} X_i = X_1 + X_2 + \cdots + X_{100}$ be the total sum of all the die rolls.

1. Compute the expected value $E(X_i)$ and variance $V(X_i)$ of X_i. *[1]*

SOLUTION. Each of the six numbers on the faces of the die has equal probability of $\frac{1}{6}$ of coming up. By definition,

$$E(X_i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

and

$$E\left(X_i^2\right) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6} \approx 15.1667\,,$$

so

$$V(X_i) = E\left(X_i^2\right) - \left[E(X_i)\right]^2 = \frac{91}{6} - \left[\frac{7}{2}\right]^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12} \approx 2.9167 \quad \square$$

2. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[1]*

SOLUTION. Since $X = X_1 + X_2 + \cdots + X_{100}$, we have

$$E(X) = E(X_1 + X_2 + \cdots + X_{100}) = E(X_1) + E(X_2) + \cdots + E(X_{100}) = 100 \cdot 3.5 = 350$$

and, since we also have that the X_i are all independent,

$$\begin{aligned} V(X) &= V(X_1 + X_2 + \cdots + X_{100}) = V(X_1) + V(X_2) + \cdots + V(X_{100}) \\ &= 100 \cdot \frac{35}{12} = \frac{875}{3} \approx 291.6667 \quad \square \end{aligned}$$

3. Approximate $P(400 \le X \le 500)$ using the standard normal distribution. *[3]*

SOLUTION. The standard deviation of X is $\sigma_X = \sqrt{V(X)} = \sqrt{\frac{875}{3}} \approx 17.0783$. We compute the "$Z$-score" of X, *i.e.* we let $Z = \frac{X - E(X)}{\sigma_X} \approx \frac{X - 350}{17.0783}$. Then the Central Limit Theorem suggests that Z has something close to a standard normal distribution, so

$$\begin{aligned} P(400 \le X \le 500) &\approx P\left(\frac{400-350}{17.0783} \le \frac{X-350}{17.0783} \le \frac{500-350}{17.0783}\right) \approx P(2.93 \le Z \le 8.78) \\ &\approx P(Z \le 8.78) - P(Z < 2.93) = 1 - 0.9983 = 0.0017 \end{aligned}$$

Note that 8.78 is off the charts on the cumulative standard normal table, so $P(Z \le 8.78)$ is close enough to 1 to make no significant difference here, and that according to the table, $P(Z < 2.93) \approx 0.9983$. ■