Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2020 (S62)

Solutions to Quiz #5
Tuesday, 21 July

The continuous random variable X has the following probability density function:

$$f(x) = \frac{1}{2}e^{-|x-1|}$$

1. Verify that $f(x)$ is a valid probability density function. *[2]*

SOLUTION. We need to check the three conditions for being a probability density function.

First, $f(x) = \frac{1}{2}e^{-|x-1|} \geq 0$ for all x because $e^c > 0$ for every real number c.

Second, since $f(x)$ is a composition of the functions $g(x) = -|x-1|$ and $h(y) = \frac{1}{2}e^y$, which are both defined and continuous for all x and y, respectively, $f(x) = h(g(x)) = \frac{1}{2}e^{-|x-1|}$ is also defined and continuous, and hence integrable, for all x.

Third, we check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{\infty} \frac{1}{2}e^{-|x-1|}\,dx = \int_{-\infty}^{1} \frac{1}{2}e^{-(1-x)}\,dx + \int_{1}^{\infty} \frac{1}{2}e^{-(x-1)}\,dx \\
&= \frac{1}{2}\int_{-\infty}^{1} e^{x-1}\,dx + \frac{1}{2}\int_{1}^{\infty} e^{1-x}\,dx
\end{aligned}$$

Substitute $u = x-1$ and $w = 1-x$, so $du = dx$ and $dw = (-1)\,dx$, and $dx = (-1)\,dw$, and change limits: $\begin{matrix} x & -\infty & 1 \\ u & -\infty & 0 \end{matrix}$ & $\begin{matrix} x & 1 & \infty \\ w & 0 & -\infty \end{matrix}$

$$\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{0} e^u\,du + \frac{1}{2}\int_{0}^{-\infty} e^w(-1)\,dw = \frac{1}{2}e^u\Big|_{-\infty}^{0} + \frac{-1}{2}e^w\Big|_{0}^{-\infty} \\
&= \frac{1}{2}e^0 - \frac{1}{2}e^{-\infty} + \frac{-1}{2}e^{-\infty} - \frac{-1}{2}e^0 = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = 1
\end{aligned}$$

Since $f(x)$ satisfies all three conditions, it is a valid probability density function. $\square$

2. Compute the expected value $E(X)$ of X. *[1.5]*

SOLUTION. *(Using calculus.)* By definition,

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{\infty} x\cdot\frac{1}{2}e^{-|x-1|}\,dx = \int_{-\infty}^{1} \frac{1}{2}xe^{-(1-x)}\,dx + \int_{1}^{\infty} \frac{1}{2}xe^{-(x-1)}\,dx \\
&= \frac{1}{2}\int_{-\infty}^{1} xe^{x-1}\,dx + \frac{1}{2}\int_{1}^{\infty} xe^{1-x}\,dx
\end{aligned}$$

As in the solution above to question **1**, substitute $u = x - 1$ and $w = 1 - x$, so $du = dx$ and $dw = (-1)\,dx$, and $dx = (-1)\,dw$, and change limits: $\begin{array}{ccc} x & -\infty & 1 \\ u & -\infty & 0 \end{array}$ & $\begin{array}{ccc} x & 1 & \infty \\ w & 0 & -\infty \end{array}$ We then have $x = u + 1$ and $x = 1 - w$ as well, so:

$$\begin{aligned} E(X) &= \frac{1}{2}\int_{-\infty}^{1} xe^{x-1}\,dx + \frac{1}{2}\int_{1}^{\infty} xe^{1-x}\,dx \\ &= \frac{1}{2}\int_{-\infty}^{0} (u+1)e^{u}\,du + \frac{1}{2}\int_{0}^{-\infty} (1-w)e^{w}(-1)\,dw \\ &= \frac{1}{2}\int_{-\infty}^{0} ue^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} e^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} (1-w)e^{w}\,dw \\ &= \frac{1}{2}\int_{-\infty}^{0} ue^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} e^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} e^{w}\,dw - \frac{1}{2}\int_{-\infty}^{0} we^{w}\,dw \end{aligned}$$

Since $\int_{-\infty}^{0} e^{u}\,du = \int_{-\infty}^{0} e^{w}\,dw$ and $\int_{-\infty}^{0} ue^{u}\,du = \int_{-\infty}^{0} we^{w}\,dw$, it follows that:

$$\begin{aligned} E(X) &= \frac{1}{2}\int_{-\infty}^{0} ue^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} e^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} e^{w}\,dw - \frac{1}{2}\int_{-\infty}^{0} e^{w}\,dw \\ &= \int_{-\infty}^{0} e^{u}\,du = e^{u}\big|_{-\infty}^{0} = e^{0} - e^{-\infty} = 1 - 0 = 1 \qquad \square \end{aligned}$$

Solution. *(Without calculus.)* Observe that for any real number a, we have

$$f(1+a) = \frac{1}{2}e^{-||(1+a)-1||} = \frac{1}{2}e^{-|a|} = \frac{1}{2}e^{-|-a|} = \frac{1}{2}e^{-|(1-a)-1|} = f(1-a)\,.$$

It follows that the graph of $f(x)$ is symmetric about the line $x = 1$, and so, assuming that $E(X)$ is defined at all, we must have $E(X) = 1$. $\square$

3. Compute the variance $V(X)$ and standard deviation σ_X of X. *[1.5]*

Solution. By definition, $V(X) = E\left(X^2\right) - [E(X)]^2$. We worked out $E(X) = 1$ in solving question **2** above, so we still need to compute $E\left(X^2\right)$. By definition,

$$\begin{aligned} E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2}e^{-|x-1|}\,dx \\ &= \int_{-\infty}^{1} \frac{1}{2}x^2 e^{-(1-x)}\,dx + \int_{1}^{\infty} \frac{1}{2}x^2 e^{-(x-1)}\,dx \\ &= \frac{1}{2}\int_{-\infty}^{1} x^2 e^{x-1}\,dx + \frac{1}{2}\int_{1}^{\infty} x^2 e^{1-x}\,dx \end{aligned}$$

As in the solutions above to questions **1** and **2**, substitute $u = x - 1$ and $w = 1 - x$, so $du = dx$ and $dw = (-1)\,dx$, and $dx = (-1)\,dw$, and change limits: $\begin{matrix} x & -\infty & 1 \\ u & -\infty & 0 \end{matrix}$ & $\begin{matrix} x & 1 & \infty \\ w & 0 & -\infty \end{matrix}$ We then have $x = u + 1$ and $x = 1 - w$ as well, so:

$$\begin{aligned} E\left(X^2\right) &= \frac{1}{2}\int_{-\infty}^{1} x^2 e^{x-1}\,dx + \frac{1}{2}\int_{1}^{\infty} x^2 e^{1-x}\,dx \\ &= \frac{1}{2}\int_{-\infty}^{0} (u+1)^2 e^u\,du + \frac{1}{2}\int_{0}^{-\infty} (1-w)^2 e^w (-1)\,dw \\ &= \frac{1}{2}\int_{-\infty}^{0} \left(u^2 + 2u + 1\right) e^u\,du + \frac{1}{2}\int_{-\infty}^{0} \left(1 - 2w + w^2\right) e^w\,dw \\ &= \frac{1}{2}\int_{-\infty}^{0} u^2 e^u\,du + \frac{1}{2}\int_{-\infty}^{0} 2ue^u\,du + \frac{1}{2}\int_{-\infty}^{0} e^u\,du \\ &\quad + \frac{1}{2}\int_{-\infty}^{0} e^w\,dw - \frac{1}{2}\int_{-\infty}^{0} 2we^w\,dw + \frac{1}{2}\int_{-\infty}^{0} w^2 e^w\,dw \end{aligned}$$

Since we have that $\int_{-\infty}^{0} e^u\,du = \int_{-\infty}^{0} e^w\,dw$, $\int_{-\infty}^{0} ue^u\,du = \int_{-\infty}^{0} we^w\,dw$, and also that $\int_{-\infty}^{0} u^2 e^u\,du = \int_{-\infty}^{0} w^2 e^w\,dw$, we now have:

$$\begin{aligned} E\left(X^2\right) &= \frac{1}{2}\int_{-\infty}^{0} u^2 e^u\,du + \frac{1}{2}\int_{-\infty}^{0} 2ue^u\,du + \frac{1}{2}\int_{-\infty}^{0} e^u\,du \\ &\quad + \frac{1}{2}\int_{-\infty}^{0} e^w\,dw - \frac{1}{2}\int_{-\infty}^{0} 2we^w\,dw + \frac{1}{2}\int_{-\infty}^{0} w^2 e^w\,dw \\ &= \int_{-\infty}^{0} u^2 e^u\,du + \int_{-\infty}^{0} e^u\,du \end{aligned}$$

We work out these integrals separately, the latter first:

$$\int_{-\infty}^{0} e^u\,du = e^u\Big|_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1$$

Oh, wait! We could have skipped that because we already did it in solving question **2** ...

To work out $\int_{-\infty}^{0} u^2 e^u\,du$ we resort to integration by parts, with $s = u^2$ and $t' = e^u$, so $s' = 2u$ and $t = e^u$. This gives:

$$\begin{aligned} \int_{-\infty}^{0} u^2 e^u\,du &= u^2 e^u\Big|_{-\infty}^{0} - \int_{-\infty}^{0} 2ue^u\,du = 0^2 e^0 - (-\infty)^2 e^{-\infty} - 2\int_{-\infty}^{0} ue^u\,du \\ &= 0 - 0 - 2\int_{-\infty}^{0} ue^u\,du = -2\int_{-\infty}^{0} ue^u\,du \end{aligned}$$

Technically, we should evaluate a limit to work out "$(-\infty)^2 e^{-\infty}$", but knowing that exponential functions dominate polunomials tells us that the 0 that e^u tends to as $u \to -\infty$ wins over the ∞ that u^2 tends to at the same time.

It remains to evaluate $\int_{-\infty}^{0} ue^u \, du$. We use parts again, this time with $p = u$ and $q' = e^u$, so $p' = 1$ and $q = e^u$. Then

$$\int_{-\infty}^{0} ue^u \, du = ue^u\big|_{-\infty}^{0} - \int_{-\infty}^{0} e^u \, du = 0e^0 - (-\infty)e^{\infty} - 1 = 0 - 0 - 1\,,$$

where we once again exploit the fact that exponentials dominate polynomials to avoid computing a limit, as well as take advantage of having computed a certain integral once – er, twice – before. Putting all these pieces together, we get that:

$$E\left(X^2\right) = \int_{-\infty}^{0} u^2 e^u \, du + \int_{-\infty}^{0} e^u \, du = -2\int_{-\infty}^{0} ue^u \, du + 1 = -2 \cdot (-1) + 1 = 3$$

Thus $V(X) = E\left(X^2\right) - [E(X)]^2 = 3 - 1^2 = 2$ and $\sigma_X = \sqrt{V(X)} = \sqrt{2}$. ■