Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2020 (S62)

Solutions to *the even more corrected* **Quiz #4**
~~*Tuesday, 14 July.*~~ *Wednesday, 15 July*

Suppose the continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} \frac{3}{4}\left(1-x^2\right) & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases}$$

1. Verify that $f(x)$ is a valid probability density function *[2.5]*

Solution. We need to check three conditions:

i. $f(x)$ is integrable because it is continuous: when $|x| \geq 1$, $f(x)$ is constant, hence continuous; when $-1 < x < 1$, $f(x) = \frac{3}{4}\left(1-x^2\right)$ is a polynomial, hence continuous; at $x = \pm 1$, $\frac{3}{4}\left(1-(\pm 1)^2\right) = \frac{3}{4}(1-1) = 0$, so the two definitions are equal at the points the transition between them happens, so $f(x)$ is coninuous everywhere.

ii. $f(x) = 0 \geq 0$ when $|x| > 0$, and when $-1 < x < 1$ we have $x^2 < 1$ and $1 - x^2 > 0$, making $f(x) = \frac{3}{4}\left(1-x^2\right) \geq 0$. Thus $f(x) \geq 0$ for all x.

iii. We need to check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{4}\left(1-x^2\right)\,dx + \int_{1}^{\infty} 0\,dx \\
&= 0 + \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} + 0 = \frac{3}{4}\left(1 - \frac{1^3}{3}\right) - \frac{3}{4}\left((-1) - \frac{(-1)^3}{3}\right) \\
&= \frac{3}{4}\cdot\frac{2}{3} - \frac{3}{4}\left(-\frac{2}{3}\right) = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

Thus $f(x)$ is indeed a valid probability density function. $\square$

2. Compute the expected value $E(X)$ of X. *[1]*

Solution. We plug our density into the definition of expected value for continuous random variables and integrate away:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{-1} x \cdot 0\,dx + \int_{-1}^{1} x \cdot \frac{3}{4}\left(1-x^2\right)\,dx + \int_{1}^{\infty} x \cdot 0\,dx \\
&= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{4}\left(x-x^3\right)\,dx + \int_{1}^{\infty} 0\,dx \\
&= 0 + \frac{3}{4}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{-1}^{1} + 0 = \frac{3}{4}\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \frac{3}{4}\left(\frac{(-1)^2}{2} - \frac{(-1)^4}{4}\right) \\
&= \frac{3}{4}\cdot\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{3}{4}\cdot\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{4}\cdot\frac{1}{4} - \frac{3}{4}\cdot\frac{1}{4} = 0 \qquad \square
\end{aligned}$$

3. How could you have computed $E(X)$ without working out the relevant integral? *[0.5]*

SOLUTION. Since $f(x)$ is an even function, *i.e.* $f(-x) = f(x)$ for all x, $xf(x)$ is an odd function, *i.e.* $(-x)f(-x) = -xf(x)$ for all x. The integral of any odd function over an interval symmetric about $x = 0$ has to be 0 because any area below (respectively, above) the graph to the right of 0 is cancelled out by the symmetric area above (respectively, below) the graph to the left of 0.

Alternatively, one could observe that the graph of $f(x)$ is symmetric about the line $x = 0$ (which is what being even boils down to), so the "weighted by $f(x)$ average value of x" ought to be $x = 0$. $\square$

4. Compute the variance $V(X)$ and standard deviation σ_X of X. *[1]*

SOLUTION. By definition, $V(X) = E\left(X^2\right) - [E(X)]^2$. We computed $E(X)$ in **2**, but we still need to work out $E\left(X^2\right)$.

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{-1} x^2 \cdot 0\,dx + \int_{-1}^{1} x^2 \cdot \frac{3}{4}\left(1 - x^2\right)\,dx + \int_{1}^{\infty} x^2 \cdot 0\,dx \\
&= 0 + \int_{-1}^{1} \frac{3}{4}\left(x^2 - x^4\right)\,dx + 0 = \frac{3}{4}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{-1}^{1} \\
&= \frac{3}{4}\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \frac{3}{4}\left(\frac{(-1)^3}{3} - \frac{(-1)^5}{5}\right) = \frac{3}{4}\left(\frac{1}{3} - \frac{1}{5}\right) - \frac{3}{4}\left(-\frac{1}{3} - \left(\frac{1}{5}\right)\right) \\
&= \frac{3}{4} \cdot \frac{2}{15} - \frac{3}{4}\left(-\frac{2}{15}\right) = \frac{6}{60} + \frac{6}{60} = \frac{12}{60} = \frac{1}{5}
\end{aligned}$$

It follows that the variance of X is $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{1}{5} - 0^2 = \frac{1}{5}$ and its standard deviation is $\sigma_X = \sqrt{V(X)} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$. $\blacksquare$