Mathematics 1550H – Probability I: Introduction to Probability

TRENT UNIVERSITY, Summer 2020 (S62)

Solution to Quiz #3

Recall that a standard 52-card deck has four *suits*, $\heartsuit$, $\diamondsuit$, $\clubsuit$, and $\spadesuit$, each of which has 13 cards, one each of the following *kinds*, A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, and 2. A hand of seven (7) cards is drawn at random from such a deck. (This means that you get the cards as a group, in no particular order, and with no possible way of getting the same card twice in the hand.) Find the probability that the hand ...

1. ... is a *flush*, *i.e.* all the cards in the hand are from the same suit. *[1]*

SOLUTION. There are $\binom{52}{7} = \frac{52!}{7!(52-7)!} = \frac{52!}{7!45!} = 133784560$ possible seven-card hands, any particular one of which is as likely as any other. There are $\binom{4}{1} = 4$ ways to pick a suit and $\binom{13}{7} = 1716$ ways to get a hand of seven cards from the chosen suit. It follows that there are $\binom{4}{1}\binom{13}{7} = 4 \cdot 1716 = 6864$ hands which are flushes, and thus the probability of getting a flush is:

$$P(\text{flush}) = \frac{\#\text{ flush hands}}{\#\text{ possible hands}} = \frac{\binom{4}{1}\binom{13}{7}}{\binom{52}{7}} = \frac{6864}{133784560} \approx 0.00005 \qquad \square$$

NOTE: We'll be using the fact that there are $\binom{52}{7} = 133784560$ possible seven-card hands without further comment in the remaining solutions.

2. ... has four cards of the same kind. *[1]*

SOLUTION. There are $\binom{13}{1} = 13$ ways to pick a kind and $\binom{4}{4} = 1$ way to choose four cards of the chosen kind. Having done so, there are $\binom{52-4}{3} = \binom{48}{3} = 17296$ ways to choose the remaining three cards in the hand from the remaining 48 cards in the deck. It follows that the number of hands with "four of a kind" is $\binom{13}{1}\binom{4}{1}\binom{48}{3} = 13 \cdot 1 \cdot 17296 = 224848$, and thus theprobability of getting a hand with four of a kind is:

$$\begin{aligned} P(\text{four of a kind}) &= \frac{\#\text{ hands with four of a kind}}{\#\text{ possible hands}} \\ &= \frac{\binom{13}{1}\binom{4}{1}\binom{48}{3}}{\binom{52}{7}} = \frac{224848}{133784560} \approx 0.00168 \qquad \square \end{aligned}$$

3. ... has exactly three cards of one kind, two cards of another kind, and two cards of yet another kind. *[1]*

SOLUTION. There are $\binom{13}{1} = 13$ ways to pick the kind from which three cards will be chosen, $\binom{4}{3} = 4$ ways to choose the three cards from that kind, $\binom{12}{2} = 66$ ways to pick the two kinds from which two cards each will be chosen, and $\binom{4}{2} = 6$ ways to choose two cards of that kind for each of the selected kinds. (Note that three of a kind and two pairs gives us a seven-card hand, so there are no more cards to pick.) It follows that the number of such hands is $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}\binom{4}{2} = 13 \cdot 4 \cdot 66 \cdot 6 \cdot 6 = 123552$, and thus the probability of getting a hand with exactly three cards of one kind, two cards of another kind, and two cards of yet another kind, is:

$$\begin{aligned} P(3 \text{ of one kind, 2 each of two other kinds}) &= \frac{\# \text{ such hands}}{\# \text{ possible hands}} \\ &= \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}\binom{4}{2}}{\binom{52}{7}} \\ &= \frac{123552}{133784560} \approx 0.00092 \qquad \square \end{aligned}$$

4. ... has cards of seven different kinds. *[1]*

SOLUTION. This would be a hand in which each card is of a different kind. There are $\binom{13}{7} = 1716$ ways to pick seven different kinds from thirteen different kinds. For each kind, there are $\binom{4}{1} = 4$ ways to choose a single card of that kind. It follows that the number of such hands is $\binom{13}{7}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = \binom{13}{7}\binom{4}{1}^7 = 1716 \cdot 4^7 = 28114944$, and thus the probability of getting a hand with cards of seven different kinds is:

$$\begin{aligned} P(7 \text{ different kinds}) &= \frac{\# \text{ such hands}}{\# \text{ possible hands}} = \frac{\binom{13}{7}\binom{4}{1}^7}{\binom{52}{7}} \\ &= \frac{28114944}{133784560} \approx 0.21015 \qquad \square \end{aligned}$$

5. ... is a *straight*, *i.e.* a set of cards that can be arranged to be consecutive with no gaps in the sequence $A\,K\,Q\,J\,10\,9\,8\,7\,6\,5\,4\,3\,2$, where we allow the sequence to wrap around the end. (So $3\,2\,A\,K\,Q\,J\,10$ would count as a straight, for example.) *[1]*

SOLUTION. A sequence of seven consecutive kinds is completely determined by the first card in the sequence. For example, if the first card in the sequence is 5, the consecutive order given forces the corresponding sequence to be $5\,4\,3\,2\,A\,K\,Q$. Since there are only thirteen kinds, there are only $\binom{13}{1} = 13$ possible consecutive sequences of seven kinds. However, there are $\binom{4}{1} = 4$ possible choices of a card of each kind. It follows that the number of straights is $\binom{13}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = \binom{13}{1}\binom{4}{1}^7 = 13 \cdot 4^7 = 212992$, and thus the probability of getting a straight is:

$$P(\text{straight}) = \frac{\# \text{ straights}}{\# \text{ possible hands}} = \frac{\binom{13}{1}\binom{4}{1}^7}{\binom{52}{7}} = \frac{212992}{133784560} \approx 0.00159 \qquad \blacksquare$$