Mathematics 1550H – Probability I: Introduction to Probability

TRENT UNIVERSITY, Summer 2020 (S62)

Solutions to Quiz #1

Consider the following experiment. A box contains six marbles: one purple marble, two identical green marbles, and three identical yellow marbles. One marble is drawn at random from the box, after which a second marble is drawn at random from the box. The first marble drawn is *not* put back in the box before the second marble is drawn. In each draw every marble then in the box has as good a chance of being drawn as any other.

1. What is an appropriate sample space for this experiment? *[1]*

SOLUTION. You can't tell the marbles of the same colour apart, so the most one record about the outcome of the experiment is the colour of the marbles and the order the colours came in. If we denote the purple marble by P, a green marble by G, and a yellow marble by Y, the sample space of possible outcomes is:

$$\Omega = \{ PG, PY, GP, YP, GG, GY, YG, YY \}$$

Note that there is no PP outcome because there is only one purple marble and it is not replaced if it is drawn first from the box. □

2. What is an appropriate probability distribution function for this experiment? *[2]*

SOLUTION. *(The verbose brute force approach.)* As usual, we'll denote the probability distribution function by m. We'll proceed to work out the probability of each outcome one by one:

First, consider the outcome PG. Since there are six marbles in the box on the first draw and only one of them is purple, you have a probability of $\frac{1}{6}$ of getting the purple marble on the first draw. If you have obtained the purple marble on the first draw, there are five marbles remaining in the box, two green and three yellow, so there is a probability of $\frac{2}{5}$ of getting a green marble on the second draw. The probability of $\frac{1}{6}$ on the first draw followed by a probability of $\frac{2}{5}$ on the second draw means that the probability of getting a purple marble first and a green marble second is $\frac{2}{5}$ of $\frac{1}{6}$, so $m(PG) = \frac{1}{6} \cdot \frac{2}{5} = \frac{2}{30}$.

Second, consider the outcome PY. Since there are six marbles in the box on the first draw and only one of them is purple, you have a probability of $\frac{1}{6}$ of getting the purple marble on the first draw. If you have obtained the purple marble on the first draw, there are five marbles remaining in the box, two green and three yellow, so there is a probability of $\frac{3}{5}$ of getting a yellow marble on the second draw. The probability of $\frac{1}{6}$ on the first draw followed by a probability of $\frac{3}{5}$ on the second draw means that the probability of getting a purple marble first and a yellow marble second is $\frac{3}{5}$ of $\frac{1}{6}$, so $m(PG) = \frac{1}{6} \cdot \frac{3}{5} = \frac{3}{30}$.

Third, consider the outcome GP. Since there are six marbles in the box on the first draw and two of them are green, you have a probability of $\frac{2}{6}$ of getting a green marble on the first draw. If you have obtained a green marble on the first draw, there are five marbles remaining in the box, one purple, one green, and three yellow, so there is a probability of $\frac{1}{5}$ of getting the purple marble on the second draw. The probability of $\frac{2}{6}$ on the first draw followed by a probability of $\frac{1}{5}$ on the second draw means that the probability of getting a green marble first and a purple marble second is $\frac{1}{5}$ of $\frac{2}{6}$, so $m(PG) = \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30}$.

Fourth, consider the outcome YP. Since there are six marbles in the box on the first draw and three of them are yellow, you have a probability of $\frac{3}{6}$ of getting a green marble on the first draw. If you have obtained a yellow marble on the first draw, there are five marbles remaining in the box, one purple, two green, and two yellow, so there is a probability of $\frac{1}{5}$ of getting the purple marble on the second draw. The probability of $\frac{3}{6}$ on the first draw followed by a probability of $\frac{1}{5}$ on the second draw means that the probability of getting a yellow marble first and a purple marble second is $\frac{1}{5}$ of $\frac{3}{6}$, so $m(PG) = \frac{3}{6} \cdot \frac{1}{5} = \frac{3}{30}$.

Fifth, consider the outcome GG. Since there are six marbles in the box on the first draw and two of them are green, you have a probability of $\frac{2}{6}$ of getting a green marble on the first draw. If you have obtained a green marble on the first draw, there are five marbles remaining in the box, one purple, one green, and three yellow, so there is a probability of $\frac{1}{5}$ of getting the green marble on the second draw. The probability of $\frac{2}{6}$ on the first draw followed by a probability of $\frac{1}{5}$ on the second draw means that the probability of getting a green marble first and the other green marble second is $\frac{1}{5}$ of $\frac{2}{6}$, so $m(GG) = \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30}$.

Sixth, consider the outcome GY. Since there are six marbles in the box on the first draw and two of them are green, you have a probability of $\frac{2}{6}$ of getting a green marble on the first draw. If you have obtained a green marble on the first draw, there are five marbles remaining in the box, one purple, one green, and three yellow, so there is a probability of $\frac{3}{5}$ of getting a yellow marble on the second draw. The probability of $\frac{2}{6}$ on the first draw followed by a probability of $\frac{3}{5}$ on the second draw means that the probability of getting a green marble first and a yellow marble second is $\frac{3}{5}$ of $\frac{2}{6}$, so $m(GY) = \frac{2}{6} \cdot \frac{3}{5} = \frac{6}{30}$.

Seventh, consider the outcome YG. Since there are six marbles in the box on the first draw and three of them are yellow, you have a probability of $\frac{3}{6}$ of getting a yellow marble on the first draw. If you have obtained a yellow marble on the first draw, there are five marbles remaining in the box, one purple, two green, and two yellow, so there is a probability of $\frac{2}{5}$ of getting a green marble on the second draw. The probability of $\frac{3}{6}$ on the first draw followed by a probability of $\frac{2}{5}$ on the second draw means that the probability of getting a yellow marble first and a green marble second is $\frac{2}{5}$ of $\frac{3}{6}$, so $m(YG) = \frac{3}{6} \cdot \frac{2}{5} = \frac{6}{30}$.

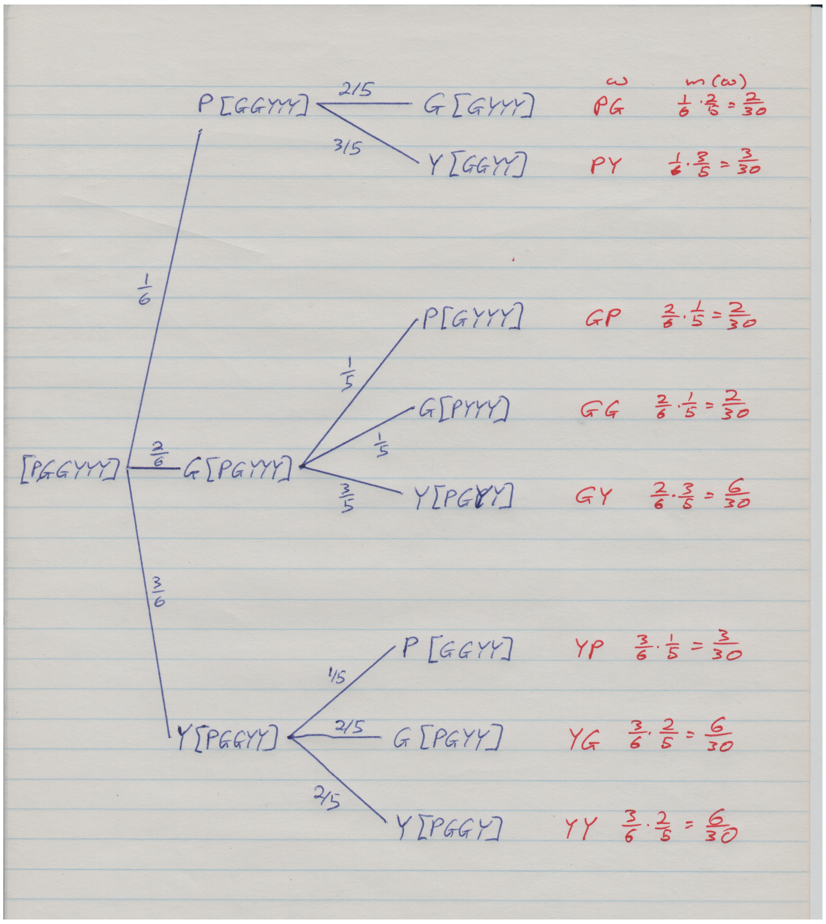
Eighth, consider the outcome YY. Since there are six marbles in the box on the first draw and three of them are yellow, you have a probability of $\frac{3}{6}$ of getting a yellow marble on the first draw. If you have obtained a yellow marble on the first draw, there are five marbles remaining in the box, one purple, two green, and two yellow, so there is a probability of $\frac{2}{5}$ of getting a yellow marble on the second draw. The probability of $\frac{3}{6}$ on the first draw followed by a probability of $\frac{2}{5}$ on the second draw means that the probability of getting a yellow marble first and a yellow marble second is $\frac{2}{5}$ of $\frac{3}{6}$, so $m(YY) = \frac{3}{6} \cdot \frac{2}{5} = \frac{6}{30}$.

As a sanity check, we'll add up the probabilities of all the outcomes and hope that sum is 1, as required for a probability distribution function.

$$\begin{aligned}\sum_{\omega\in\Omega} m(\omega) &= m(PG) + m(PY) + m(GP) + m(YP) \\ &\quad + m(GG) + m(GY) + m(YG) + m(YY) \\ &= \frac{2}{30} + \frac{3}{30} + \frac{2}{30} + \frac{3}{30} + \frac{2}{30} + \frac{6}{30} + \frac{6}{30} + \frac{6}{30} \\ &= \frac{2+3+2+3+2+6+6+6}{30} = \frac{30}{30} = 1\end{aligned}$$

They do add up to 1. Whew! □

SOLUTION. *(Using a tree diagram with extras.)* We'll use a tree diagram to pick apart the experiment. We'll record a little extra information on the tree, namely what marbles are remaining in the box at each stage, not just what marble has been drawn. This makes it easy to work out what the probabilities of selecting one of the remaining marbles is.



For the obvious sanity check, see the previous solution. □

3. Let A be the event that the two marbles drawn were of different colours. Compute the probability $P(A)$ of the event A. *[2]*

SOLUTION. Considering the sample space $\Omega = \{ PG, PY, GP, YP, GG, GY, YG, YY \}$, it's not hard to see that the event consisting of those outcomes in which the colours of the marbles drawn are different is $A = \{ PG, PY, GP, YP, GY, YG \}$. The probability of A is therefore:

$$
\begin{aligned}
P(A) &= \sum_{\omega \in \Omega} m(\omega) = m(PG) + m(PY) + m(GP) + m(YP) + m(GY) + m(YG) \\
&= \frac{2}{30} + \frac{3}{30} + \frac{2}{30} + \frac{3}{30} + \frac{6}{30} + \frac{6}{30} = \frac{2+3+2+3+6+6}{30} = \frac{22}{30} = \frac{11}{15} \approx 0.7333
\end{aligned}
$$

... and that's all, folks! ■