Mathematics 1550H – Introduction to probability
Trent University, Summer 2020 (S62)

Take-Home Final Examination
Released at noon on Wednesday, 29 July, 2020.
Due by noon on Saturday, 1 August, 2020.

Instructions

- You may consult your notes, handouts, and textbook from this course and any other math courses you have taken or are taking now. You may also use a calculator. However, you may not consult any other source, or give or receive any other aid, except for asking the instructor to clarify instructions or questions.
- Please submit an electronic copy of your solutions, preferably as a single pdf (a scan of handwritten solutions should be fine), via the Assignment module on Blackboard. If that doesn't work, please email your solutions to the intructor. *Show all your work!*
- Do part ♡ and, if you wish, part ♣.

Part ♡. Do any eight (8) of **1**–**10**.

1. A fair coin is tossed three times. Let A be the event that there are at least two heads in the three tosses and let B be the event that there are exactly two heads among the three tosses.

a. Draw the complete tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Compute $P(A)$, $P(B)$, $P(A|B)$, and $P(B|A)$. *[7]*

2. Let U be a continuous random variable with the following probability density function:

$$g(t) = \begin{cases} 1+t & -1 \leq t \leq 0 \\ 1-t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $g(t)$ is indeed a probability density function. *[5]*

b. Compute the expected value, $E(U)$, and variance, $V(U)$, of U. *[10]*

3. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck. Let A be the event that the hand includes four cards of the same kind, and let B be the event that at least two of the cards in the hand are of the same kind.

a. Compute $P(A)$. *[5]* **b.** Compute $P(B)$. *[5]* **c.** Compute $P(A|B)$. *[5]*

4. Suppose Z and X are continuous random variables such that Z has a standard normal distribution and $X = 5Z + 10$.

a. Compute $P(7 \leq X \leq 17)$. *[6]*

b. What are the expected value $E(X)$ and variance $V(X)$ of X? *[6]*

c. What kind of distribution does X have? *[3]*

[More of Part ♡ on page 2.]

5. Suppose X is a discrete random variable that has a geometric distribution with $p = \frac{1}{2}$.

a. Compute $P(X \geq 6)$. *[5]*

b. Use Markov's Inequality to estimate $P(X \geq 6)$. *[5]*

c. Use Chebyshev's Inequality to estimate $P(X \geq 6)$. *[5]*

6. Let $g(t) = \begin{cases} 2te^{-t^2} & t \geq 0 \\ 0 & t < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(t)$ is indeed a probability density function. *[8]*

b. Find the *median* of X, *i.e.* the number m such that $P(X \leq m) = \frac{1}{2} = 0.5$. *[7]*

7. A jar contains 6 white beads and 3 black beads. Beads are chosen randomly from the jar one at a time until the third time a black bead turns up.

a. Suppose that each bead is replaced before the next is chosen. How many beads should you expect to be chosen in the course of the experiment? *[5]*

b. Suppose that if a bead is white, it is *not* replaced before the next bead is chosen, but if it is black, it *is* replaced before the next is chosen. How many beads should you expect to be chosen in the course of the experiment? *[10]*

8. For each $i = 1, 2, \ldots, 10$, X_i is a random variable that gives 0 or 1 if the ith toss of a fair coin came up T or H, respectively. Let $X = X_1 + X_2 + \cdots + X_{10}$.

a. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[5]*

b. What is the probability function of X? *[10]*

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$Y \backslash X$	-1	0	1
1	0.1	0.1	0.1
3	0	0.2	0.1
4	0.2	0.1	0.1

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\text{Cov}(X, Y)$ of X and Y. *[11]*

b. Let $W = X - Y$. Compute $E(W)$ and $V(W)$. *[4]*

10. Let X be a continuous random variable with probability density function

$$h(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

a. Verify that $h(x)$ is a valid probability density function. *[7]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[8]*

[Total = 8×15 = 120]

[Part ♣ is on page 3.]

Part ♣. Bonus!

○●. There are 64 teams who play in a single elimination tournament (hence 6 rounds), and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points. *[1]*

●○. Write an original little poem about probability or mathematics in general. *[1]*

Thank you all for bearing with the course under difficult circumstances.
Enjoy the rest of the summer!