Mathematics 1550H – Introduction to probability
Trent University, Summer 2020 (S62)

Solutions to the Take-Home Final Examination

Instructions

- You may consult your notes, handouts, and textbook from this course and any other math courses you have taken or are taking now. You may also use a calculator. However, you may not consult any other source, or give or receive any other aid, except for asking the instructor to clarify instructions or questions.
- Please submit an electronic copy of your solutions, preferably as a single pdf (a scan of handwritten solutions should be fine), via the Assignment module on Blackboard. If that doesn't work, please email your solutions to the intructor. *Show all your work!*
- Do part ♡ and, if you wish, part ♣.

Part ♡. Do any eight (8) of **1**–**10**.

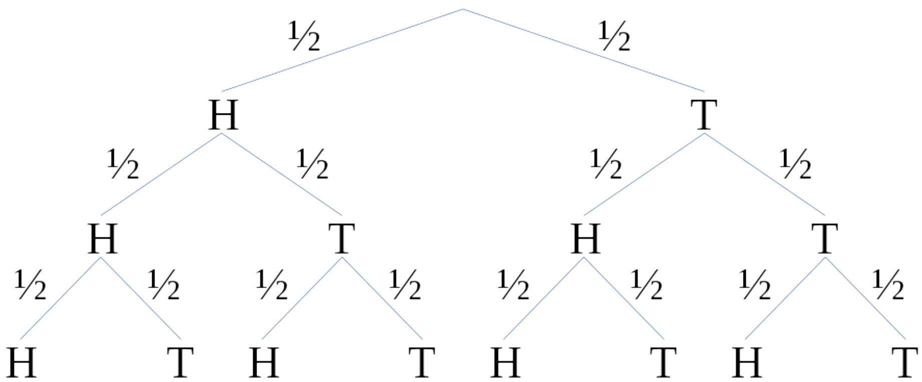
1. A fair coin is tossed three times. Let A be the event that there are at least two heads in the three tosses and let B be the event that there are exactly two heads among the three tosses.

a. Draw the complete tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Compute $P(A)$, $P(B)$, $P(A|B)$, and $P(B|A)$. *[7]*

Solutions. **a.** Here it is:



□

b. The sample space is $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Since the coin is fair, each of the eight possible outcomes is equally likely, so the probability function is given by $m(\omega) = \dfrac{1}{\#\text{ outcomes}} = \dfrac{1}{8}$ for each outcome $\omega \in \Omega$. □

c. $A = \{HHH, HHT, HTH, THH\}$ and $B = \{HHT, HTH, THH\}$; note that $B \subseteq A$. It follows that:

$$P(A) = m(HHH) + m(HHT) + m(HTH) + m(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

$$P(B) = m(HHT) + m(HTH) + m(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(A \cap B) = P(B) = \frac{3}{8} \quad (A \cap B = B \text{ because } B \subseteq A.)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{3/8} = 1 \qquad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3/8}{1/2} = \frac{3}{4} \qquad \blacksquare$$

2. Let U be a continuous random variable with the following probability density function:

$$g(t) = \begin{cases} 1+t & -1 \leq t \leq 0 \\ 1-t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $g(t)$ is indeed a probability density function. *[5]*

b. Compute the expected value, $E(U)$, and variance, $V(U)$, of U. *[10]*

SOLUTIONS. **a.** *i.* Since $1+t \geq 0$ for $-1 \leq t \leq 0$, $1-t \geq 0$ for $0 \leq t \leq 1$, and $0 \geq 0$ (for all t :-), it follows by the definition of $g(t)$ that $g(t) \geq 0$ for all t.

ii. Note that $1+t$, $1-t$, and the constant function 0 are all continuous. As $1+t=0$ when $t=-1$, $1+t=1=1-t$ when $t=0$, and $1-t=0$ when $t=1$, it follows that $g(t)$ is continuous everywhere. Since it is continuous, $g(t)$ is integrable.

iii. We check that the integral $\int_{-\infty}^{\infty} g(t)\,dt = 1$:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(t)\,dt &= \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{0} (1+t)\,dt + \int_{0}^{1} (1-t)\,dt + \int_{1}^{\infty} 0\,dt \\
&= 0 + \left(t + \frac{t^2}{2}\right)\Bigg|_{-1}^{0} + \left(t - \frac{t^2}{2}\right)\Bigg|_{0}^{1} + 0 \\
&= \left(0 + \frac{0^2}{2}\right) - \left((-1) + \frac{(-1)^2}{2}\right) + \left(1 - \frac{1^2}{2}\right) - \left(0 - \frac{0^2}{2}\right) \\
&= 0 - \left(-\frac{1}{2}\right) + \frac{1}{2} - 0 = 1
\end{aligned}$$

Since it satisfies all three conditions, $g(t)$ is a valid probability density function. □

b. By definition,

$$\begin{aligned}
E(U) &= \int_{-\infty}^{\infty} tg(t)\,dt = \int_{-\infty}^{-1} t \cdot 0\,dt + \int_{-1}^{0} t(1+t)\,dt + \int_{0}^{1} t(1-t)\,dt + \int_{1}^{\infty} t \cdot 0\,dt \\
&= 0 + \int_{-1}^{0} \left(t + t^2\right)\,dt + \int_{0}^{1} \left(t - t^2\right)\,dt + 0 = \left(\frac{t^2}{2} + \frac{t^3}{3}\right)\Bigg|_{-1}^{0} + \left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Bigg|_{0}^{1} \\
&= \left(\frac{0^2}{2} + \frac{0^3}{3}\right) - \left(\frac{(-1)^2}{2} + \frac{(-1)^3}{3}\right) + \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) \\
&= 0 - \frac{1}{6} + \frac{1}{6} - 0 = 0\,.
\end{aligned}$$

To compute $V(U) = E\left(U^2\right) - \left[E\left(U^2\right)\right]^2$, we first need to work out $E\left(U^2\right)$:

$$\begin{aligned}
E\left(U^2\right) &= \int_{-\infty}^{\infty} t^2 g(t)\,dt = \int_{-\infty}^{-1} t^2 \cdot 0\,dt + \int_{-1}^{0} t^2(1+t)\,dt + \int_{0}^{1} t^2 t(1-t)\,dt + \int_{1}^{\infty} t^2 \cdot 0\,dt \\
&= 0 + \int_{-1}^{0} \left(t^2 + t^3\right)\,dt + \int_{0}^{1} \left(t^2 - t^3\right)\,dt + 0 = \left(\frac{t^3}{3} + \frac{t^4}{4}\right)\Bigg|_{-1}^{0} + \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\Bigg|_{0}^{1} \\
&= \left(\frac{0^3}{3} + \frac{0^4}{4}\right) - \left(\frac{(-1)^3}{3} + \frac{(-1)^4}{4}\right) + \left(\frac{1^3}{3} - \frac{1^4}{4}\right) - \left(\frac{0^3}{3} - \frac{0^4}{4}\right) \\
&= 0 - \left(-\frac{1}{12}\right) + \frac{1}{12} - 0 = \frac{2}{12} = \frac{1}{6}\,.
\end{aligned}$$

It follows that $V(U) = E\left(U^2\right) - \left[E\left(U^2\right)\right]^2 = \frac{1}{6} - 0^2 = \frac{1}{6}$. ■

3. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck. Let A be the event that the hand includes four cards of the same kind, and let B be the event that at least two of the cards in the hand are of the same kind.

a. Compute $P(A)$. *[5]* **b.** Compute $P(B)$. *[5]* **c.** Compute $P(A|B)$. *[5]*

SOLUTIONS. **a.** Since every hand is as likely as any other hand,

$$P(A) = \frac{\#\text{ hands in } A}{\#\text{ hands}} = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{13 \cdot 1 \cdot 48}{2598960} = \frac{624}{2598960} \approx 0.00024\,,$$

since there are $\binom{13}{1} = 13$ ways to choose a kind, $\binom{4}{4} = 1$ way to choose all four cards of that kind, and $\binom{48}{1} = 48$ ways to choose the remaining card from the $52 - 4 = 48$ cards left in the deck. □

b. Similarly,

$$\begin{aligned}
P(B) = P\left(\bar{B}\right) &= 1 - \frac{\#\text{ hands not in } B}{\#\text{ hands}} = 1 - \frac{\#\text{ hands with cards of five different kinds}}{\#\text{ hands}} \\
&= 1 - \frac{\binom{13}{5}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 1 - \frac{1287 \cdot 4^5}{2598960} = 1 - \frac{1317888}{2598960} = \frac{1281072}{2598960} \approx 0.49292\,,
\end{aligned}$$

since there are $\binom{13}{5} = 1287$ ways to pick 5 out of 13 kinds, and $\binom{4}{1} = 4$ ways to pick 1 card out of 4 cards of each chosen kind. □

c. Since $A \subseteq B$ – having four cards of the same kind certainly means that you have at least two of the same kind – we have $A \cap B = A$. It follows that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{624/2598960}{1281072/2598960} = \frac{624}{1281072} \approx 0.00049\,. \quad ■$$

4. Suppose Z and X are continuous random variables such that Z has a standard normal distribution and $X = 5Z + 10$.

a. Compute $P(7 \leq X \leq 17)$. *[6]*

b. What are the expected value $E(X)$ and variance $V(X)$ of X? *[6]*

c. What kind of distribution does X have? *[3]*

SOLUTIONS. **a.** Referring to a cumulative standard normal table as necessary:

$$\begin{aligned} P(7 \leq X \leq 17) &= P(7 \leq 5Z + 10 \leq 17) = P(-3 \leq 5Z \leq 7) = P\left(-\frac{3}{5} \leq Z \leq \frac{7}{5}\right) \\ &= P(-0.6 \leq Z \leq 1.4) = P(Z \leq 1.4) - P(Z < -0.6) \\ &\approx 0.9192 - 0.2743 = 0.6449 \qquad \square \end{aligned}$$

b. Since Z has a standard normal distribution, $E(Z) = 0$ and $V(Z) = 1$. It follows that

$$\begin{aligned} E(X) &= E(5Z + 10) = E(5Z) + E(10) = 5E(Z) + 10 = 5 \cdot 0 + 10 = 10 \\ \text{and } V(X) &= V(5Z + 10) = V(5Z) + V(10) = 5^2V(Z) + 0 = 25 \cdot 1 + 0 = 25\,. \end{aligned}$$

Note that Z (and hence $5Z$) are independent of the constant 10 (Why?), and that the expected value and variance of any constant c are c and 0 respectively. (It's a *constant*, after all.) $\square$

c. $X = 5Z + 10$ has a normal distribution with expected value $\mu = 5$ and variance $\sigma^2 = 5^2 = 25$, *i.e.* $X \sim N(10, 5)$ if you like notation. (See page 213 in the textbook for why this is so. No need to reinvent the wheel here ...) $\blacksquare$

5. Suppose X is a discrete random variable that has a geometric distribution with $p = \frac{1}{2}$.

a. Compute $P(X \geq 6)$. *[5]*

b. Use Markov's Inequality to estimate $P(X \geq 6)$. *[5]*

c. Use Chebyshev's Inequality to estimate $P(X \geq 6)$. *[5]*

SOLUTIONS. **a.** Since $p = \frac{1}{2}$, we have $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$ too. This means that the probabiltity distribution function of of X is

$$m(k) = P(X = k) = P(\text{1st success on } k \text{ trial}) = q^{k-1}p = \left(\frac{1}{2}\right)^{k-1}\frac{1}{2} = \left(\frac{1}{2}\right)^{k}.$$

for integers $k \geq 1$. ($m(k) = 0$ otherwise.) It follows that

$$\begin{aligned} P(X \geq 6) &= 1 - P(X \leq 5) = 1 - (m(1) + m(2) + m(3) + m(4) + m(5)) \\ &= 1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right) = \frac{1}{32}. \qquad \square \end{aligned}$$

b. Since X has a geometric distribution with $p = \frac{1}{2}$, $E(X) = \frac{1}{p} = \frac{1}{1/2} = 2$. Markov's Inequality then tells us that $P(X \geq 6) \leq \dfrac{E(X)}{6} = \dfrac{2}{6} = \dfrac{1}{3}$. $\square$

c. Since X has a geometric distribution with $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$, then $E(X) = \frac{1}{p} = \frac{1}{1/2} = 2$ (as above) and $V(X) = \frac{q}{p^2} = \frac{1/2}{(1/2)^2} = \frac{1}{1/2} = 2$. Note that since $X \geq 1$, $X - 2 \geq 4 \Leftrightarrow |X - 2| \geq 4$. Chebyshev's Inequality now tells us that

$$\begin{aligned} P(X \geq 6) &= P(X - 2 \geq 4) = P\left(|X - 2| \geq 4\right) \\ &= P\left(|X - E(X)| \geq 4\right) \leq \frac{V(X)}{4^2} = \frac{2}{16} = \frac{1}{8}. \end{aligned}$$ ■

6. Let $g(t) = \begin{cases} 2te^{-t^2} & t \geq 0 \\ 0 & t < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(t)$ is indeed a probability density function. *[8]*

b. Find the *median* of X, *i.e.* the number m such that $P(X \leq m) = \frac{1}{2} = 0.5$. *[7]*

SOLUTIONS. **a.** *i.* Since $2t \geq 0$ and $e^{-t^2} > 0$ when $t \geq 0$, and $0 \geq 0$ for all $t < 0$ (and $t \geq 0$ too!), it follows by its definition that $g(t) \geq 0$ for all t.
ii. $2te^{-t^2}$ and the constant function 0 are both continuous. Since $2 \cdot 0 \cdot e^{-0^2} = 0$, $g(t)$ is also continuous at $t = 0$, where its component functions are stitched together. Thus $g(t)$ is defined and continuous everywhere, and hence is integrable. *iii.* Note that

$$\int_{-\infty}^{\infty} g(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} 2te^{-t^2}\,dt \qquad \text{Substitute } u = t^2, \text{ so } du = 2t\,dt, \text{ and change limits: } \begin{matrix} t & 0 & \infty \\ u & 0 & \infty \end{matrix}.$$

$$= 0 + \int_{0}^{\infty} e^{-u}\,du = -e^{-u}\Big|_0^\infty = \left(-e^{-\infty}\right) - \left(-e^{-0}\right) = (-0) - (-1) = 1,$$

with only a little abuse of calculus by throwing ∞ around as if it were a real number. (Fortunately, it's usually safe to do that with exponential functions.)

Since it satisfies all three conditions, $g(t)$ is a valid probability density function. □

b. We want the value of m such that $P(X \leq m) = \frac{1}{2}$. Since m must be positive (Why?),

$$P(X \leq m) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{m} 2te^{-t^2}\,dt \qquad \text{Substitute } u = t^2, \text{ so } du = 2t\,dt, \text{ and change limits: } \begin{matrix} t & 0 & \infty \\ u & 0 & m^2 \end{matrix}.$$

$$= 0 + \int_{0}^{m^2} e^{-u}\,du = -e^{-u}\Big|_0^{m^2} = \left(-e^{-m^2}\right) - \left(-e^{-0}\right) = 1 - e^{-m^2},$$

so we need to solve the equation $1 - e^{-m^2} = \frac{1}{2}$ to obtain the median m. Here goes:

$$1 - e^{-m^2} = \frac{1}{2} \implies e^{-m^2} = \frac{1}{2} \implies -m^2 = \ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2) = 0 - \ln(2) = -\ln(2)$$

$$\implies m^2 = \ln(2) \implies m = \sqrt{\ln(2)} \approx 0.8326$$ ■

7. A jar contains 6 white beads and 3 black beads. Beads are chosen randomly from the jar one at a time until the third time a black bead turns up.

a. Suppose that each bead is replaced before the next is chosen. How many beads should you expect to be chosen in the course of the experiment? *[5]*

b. Suppose that if a bead is white, it is *not* replaced before the next bead is chosen, but if it is black, it *is* replaced before the next is chosen. How many beads should you expect to be chosen in the course of the experiment? *[10]*

SOLUTIONS. **a.** Let Y be the number of beads chosen in the course of the experiment, *i.e.* the number of draws required until and including the third time a black bead is drawn. If each bead is replaced before the next is randomly chosen, then there are nine beads in total for each draw, six white and three black. Thus there is a probability of $\frac{6}{9} = \frac{2}{3}$ of choosing a white bead and a probability of $\frac{3}{9} = \frac{1}{3}$ of choosing a black bead on each draw. Choosing beads until the third time a black bead is drawn is basically repeating Bernoulli trials with a probability of $p = \frac{1}{3}$ of success until the third success. Counting beads is counting the number of trials required, so this process has a negative binomial distribution with $k = 3$, $p = \frac{1}{3}$, and $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$, so it has an expected value of $E(Y) = \frac{k}{p} = \frac{3}{1/3} = 3 \cdot 3 = 9$. □

b. This is probably the hardest question to actually complete on this exam, mainly because there aren't really any good shortcuts over using brute force. (It would be a much easier problem if it instead had black beads being removed and white beads being replaced after being drawn. Why?) Only one student who attempted this problem solved it fully. Here's to you Deckar! :-) What follows is a version of Deckar's solution.

Let X be the number of beads drawn in the course of this version of the experiment; our task is to compute $E(X)$. Since white beads do not get replaced, the minimum number of beads drawn in the course of the experiment is 3 and the maximum is $6 + 3 = 9$. The underlying sample space, with the outcomes listed in order of length and alphabetically for equal length, is

$$\Omega = \{\,\text{BBB, BBWB, BWBB, WBBB, BBWWB, BWBWB, } \dots, \text{ WWWWWWBBB}\,\}\,.$$

Since the last bead drawn in any outcome is a black one, but the there are $n - 1$ previous draws in which the previous two black beads could come up, there are $\binom{n-1}{2}$ outcomes of length n for $3 \le n \le 9$.

The chance of drawing a white bead when k white beads have already been drawn is $\frac{6-k}{9-k}$, and the chance of drawing a black bead instead in this situation is $\frac{3}{9-k}$, since white beads are not replaced when drawn but black ones are. It is important to note – since there is likely to be a temptation to assume otherwise – that the probability of an outcome does not just depend on its length. For example, BBWB and BWBB are both outcomes of length 4, but $P(\text{BBWB}) = \frac{3}{9}\cdot\frac{3}{9}\cdot\frac{6}{9}\cdot\frac{3}{8} \neq \frac{3}{9}\cdot\frac{6}{9}\cdot\frac{3}{8}\cdot\frac{3}{8} = P(\text{BWBB})$. One can, however, say that for every outcome of length 4 the probability of the white bead, when it comes up, will be $\frac{6}{9}$, and that the probability of the final black bead will be $\frac{3}{8}$. This kind of analysis let's us write out the formula for the expected value $E(X)$ in less than a page:

$$
\begin{aligned}
E(X) = {} & \frac{3}{9}\cdot\frac{3}{9}\cdot\frac{3}{9}\cdot 3 + \frac{6}{9}\cdot\frac{3}{8}\cdot\left[\frac{3}{9}\cdot\frac{3}{9}+\frac{3}{9}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{3}{8}\right]\cdot 4 \\
& + \frac{6}{9}\cdot\frac{5}{8}\cdot\frac{3}{7}\cdot\left[\frac{3}{9}\cdot\frac{3}{9}+\frac{3}{9}\cdot\frac{3}{8}+\frac{3}{9}\cdot\frac{3}{7}+\frac{3}{8}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{3}{7}+\frac{3}{7}\cdot\frac{3}{7}\right]\cdot 5 \\
& + \frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\left[\frac{3}{9}\cdot\frac{3}{9}+\frac{3}{9}\cdot\frac{3}{8}+\frac{3}{9}\cdot\frac{3}{7}+\frac{3}{9}\cdot\frac{3}{6}+\frac{3}{8}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{3}{7}\right. \\
& \qquad\qquad \left. +\frac{3}{8}\cdot\frac{3}{6}+\frac{3}{7}\cdot\frac{3}{7}+\frac{3}{7}\cdot\frac{3}{6}+\frac{3}{6}\cdot\frac{3}{6}\right]\cdot 6 \\
& + \frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\left[\frac{3}{9}\cdot\frac{3}{9}+\frac{3}{9}\cdot\frac{3}{8}+\frac{3}{9}\cdot\frac{3}{7}+\frac{3}{9}\cdot\frac{3}{6}+\frac{3}{9}\cdot\frac{3}{5}+\frac{3}{8}\cdot\frac{3}{8}\right. \\
& \quad \left. +\frac{3}{8}\cdot\frac{3}{7}+\frac{3}{8}\cdot\frac{3}{6}+\frac{3}{8}\cdot\frac{3}{5}+\frac{3}{7}\cdot\frac{3}{7}+\frac{3}{7}\cdot\frac{3}{6}+\frac{3}{7}\cdot\frac{3}{5}+\frac{3}{6}\cdot\frac{3}{6}+\frac{3}{6}\cdot\frac{3}{5}+\frac{3}{5}\cdot\frac{3}{5}\right]\cdot 7 \\
& + \frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}\cdot\left[\frac{3}{9}\cdot\frac{3}{9}+\frac{3}{9}\cdot\frac{3}{8}+\frac{3}{9}\cdot\frac{3}{7}+\frac{3}{9}\cdot\frac{3}{6}+\frac{3}{9}\cdot\frac{3}{5}+\frac{3}{9}\cdot\frac{3}{4}+\frac{3}{8}\cdot\frac{3}{8}\right. \\
& \quad +\frac{3}{8}\cdot\frac{3}{7}+\frac{3}{8}\cdot\frac{3}{6}+\frac{3}{8}\cdot\frac{3}{5}+\frac{3}{8}\cdot\frac{3}{4}+\frac{3}{7}\cdot\frac{3}{7}+\frac{3}{7}\cdot\frac{3}{6}+\frac{3}{7}\cdot\frac{3}{5}+\frac{3}{7}\cdot\frac{3}{4}+\frac{3}{6}\cdot\frac{3}{6} \\
& \quad \left. +\frac{3}{6}\cdot\frac{3}{5}+\frac{3}{6}\cdot\frac{3}{4}+\frac{3}{5}\cdot\frac{3}{5}+\frac{3}{5}\cdot\frac{3}{4}+\frac{3}{4}\cdot\frac{3}{4}\right]\cdot 8 \\
& + \frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{3}{3}\cdot\left[\frac{3}{9}\cdot\frac{3}{9}+\frac{3}{9}\cdot\frac{3}{8}+\frac{3}{9}\cdot\frac{3}{7}+\frac{3}{9}\cdot\frac{3}{6}+\frac{3}{9}\cdot\frac{3}{5}+\frac{3}{9}\cdot\frac{3}{4}\right. \\
& \quad +\frac{3}{9}\cdot\frac{3}{3}+\frac{3}{8}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{3}{7}+\frac{3}{8}\cdot\frac{3}{6}+\frac{3}{8}\cdot\frac{3}{5}+\frac{3}{8}\cdot\frac{3}{4}+\frac{3}{8}\cdot\frac{3}{3}+\frac{3}{7}\cdot\frac{3}{7}+\frac{3}{7}\cdot\frac{3}{6} \\
& \quad +\frac{3}{7}\cdot\frac{3}{5}+\frac{3}{7}\cdot\frac{3}{4}+\frac{3}{7}\cdot\frac{3}{3}+\frac{3}{6}\cdot\frac{3}{6}+\frac{3}{6}\cdot\frac{3}{5}+\frac{3}{6}\cdot\frac{3}{4}+\frac{3}{6}\cdot\frac{3}{3}+\frac{3}{5}\cdot\frac{3}{5}+\frac{3}{5}\cdot\frac{3}{4} \\
& \quad \left. +\frac{3}{5}\cdot\frac{3}{3}+\frac{3}{4}\cdot\frac{3}{4}+\frac{3}{4}\cdot\frac{3}{3}+\frac{3}{3}\cdot\frac{3}{3}\right]\cdot 9 \\
\approx {} & 0.11111 + 0.37674 + 1.20174 + 1.51229 + 1.50422 + 0.99125 = 5.69735 \qquad \blacksquare
\end{aligned}
$$

8. For each $i = 1, 2, \ldots, 10$, X_i is a random variable that gives 0 or 1 if the ith toss of a fair coin came up T or H, respectively. Let $X = X_1 + X_2 + \cdots + X_{10}$.

a. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[5]*

b. What is the probability function of X? *[10]*

SOLUTIONS. **a.** Each X_i counts the number of heads that come up in a single toss of a fair coin. Thus, for each $i = 1, 2, \ldots, 10$, we have:

$$E(X_i) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$
$$E\left(X_i^2\right) = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 0^2 = \frac{1}{2}$$
$$V(X_i) = E\left(X_i^2\right) - \left[E(X_i)\right]^2 = \frac{1}{2} - \left[\frac{1}{2}\right]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Since $X = X_1 + X_2 + \cdots + X_{10}$,

$$\begin{aligned} E(X) &= E(X_1 + X_2 + \cdots + X_{10}) = E(X_1) + E(X_2) + \cdots + E(X_{10}) \\ &= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 10 \cdot \frac{1}{2} = 5\,, \end{aligned}$$

and since the X_i are also independent,

$$\begin{aligned} V(X) &= V(X_1 + X_2 + \cdots + X_{10}) = V(X_1) + V(X_2) + \cdots + V(X_{10}) \\ &= \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{4}2 = 10 \cdot \frac{1}{4} = \frac{5}{2} = 2.5\,. \qquad \square \end{aligned}$$

b. $X = X_1 + X_2 + \cdots + X_{10}$ counts the number of heads that come up in 10 tosses of a fair coin, *i.e.* that is the number of successes in 10 Bernoulli trials with a probability $p = \frac{1}{2}$ of success on each trial. It follows that X has a binomial distribution with $n = 10$ and $p = \frac{1}{2}$, and thus $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$. This means that X must have the probability distribution function

$$m(k) = P(k \text{ successes}) = \binom{10}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{10-k} = \binom{10}{k}\left(\frac{1}{2}\right)^{10},$$

where $0 \leq k \leq 10$. ($m(k) = 0$ otherwise.) ■

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$Y \backslash X$	-1	0	1
1	0.1	0.1	0.1
3	0	0.2	0.1
4	0.2	0.1	0.1

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\mathrm{Cov}(X, Y)$ of X and Y. *[11]*

b. Let $W = X - Y$. Compute $E(W)$ and $V(W)$. *[4]*

SOLUTIONS. **a.** Here goes:

$$\begin{aligned}
E(X) &= (-1) \cdot (0.1 + 0 + 0.2) + 0(0.1 + 0.2 + 0.1) + 1(0.1 + 0.1 + 0.1) \\
&= -0.3 + 0 + 0.3 = 0 \\
E(Y) &= 1(0.1 + 0.1 + 0.1) + 3(0 + 0.2 + 0.1) + 4(0.2 + 0.1 + 0.1) \\
&= 0.3 + 0.9 + 1.6 = 2.8 \\
E\left(X^2\right) &= (-1)^2 \cdot (0.1 + 0 + 0.2) + 0^2(0.1 + 0.2 + 0.1) + 1^2(0.1 + 0.1 + 0.1) \\
&= 0.3 + 0 + 0.3 = 0.6 \\
E\left(Y^2\right) &= 1^2(0.1 + 0.1 + 0.1) + 3^2(0 + 0.2 + 0.1) + 4^2(0.2 + 0.1 + 0.1) \\
&= 0.3 + 2.7 + 6.4 = 9.4 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 0.6 - 0^2 = 0.6 \\
V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 9.4 - 2.8^2 = 1.56 \\
E(XY) &= 1 \cdot (-1) \cdot 0.1 + 1 \cdot 0 \cdot 0.1 + 1 \cdot 1 \cdot 0.1 \\
&\quad + 3 \cdot (-1) \cdot 0 + 3 \cdot 0 \cdot 0.2 + 3 \cdot 1 \cdot 0.1 \\
&\quad + 4 \cdot (-1) \cdot 0.2 + 4 \cdot 0 \cdot 0.1 + 4 \cdot 1 \cdot 0.1 \\
&= -0.1 + 0 + 0.1 + 0 + 0 + 0.3 - 0.8 + 0 + 0.4 = -0.1 \\
\mathrm{Cov}(XY) &= E(XY) - E(X) \cdot E(Y) = -0.1 - 0 \cdot 2.8 = -0.1 \qquad \square
\end{aligned}$$

b. Here goes:

$$\begin{aligned}
E(W) &= E(X - Y) = E(X) - E(Y) = 0 - 2.8 = -2.8 \\
V(W) &= V(X - Y) = V\left(X + (-1)Y\right) = V(X) + V\left((-1)Y\right) + 2\mathrm{Cov}\left(X, (-1)Y\right) \\
&= V(X) + (-1)^2 V(Y) + 2(-1)\mathrm{Cov}(X, Y) = V(X) + V(Y) - 2\mathrm{Cov}(X, Y) \\
&= 0.6 + 1.56 - 2(-0.1) = 2.36 \qquad \blacksquare
\end{aligned}$$

10. Let X be a continuous random variable with probability density function

$$h(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

a. Verify that $h(x)$ is a valid probability density function. *[7]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[8]*

SOLUTIONS. **a.** *i.* $x \geq 0$ and $e^{-x} > 0$, so $h(x) = xe^{-x} \geq 0$, when $x \geq 0$, and $h(x) = 0 \geq 0$ when $x < 0$, so $h(x) \geq 0$ for all x.

ii. xe^{-x} is continuous for $x \geq 0$ and the constant function 0 is continuous for $x < 0$. Since $0e^{-0} = 0$, it follows that $h(x)$ is also continuous at $x - 0$. It follows that $h(x)$ is continuous, and hence integrable, for all x.

iii. We will use integration by parts, with $u = x$ and $v' = e^{-x}$, so $u' = 1$ and $v = -e^{-x}$, as well as l'Hôpital's Rule, to help evaluate the integral we need to check:

$$\begin{aligned}
\int_{-\infty}^{\infty} h(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} xe^{-x}\,dx = 0 + x\left(-e^{-x}\right)\Big|_0^\infty - \int_0^\infty 1\left(-e^{-x}\right)\,dx \\
&= \lim_{a\to\infty} (-1)xe^{-x}\Big|_0^a + \int_0^\infty e^{-x}\,dx \\
&= \lim_{a\to\infty} \left[(-1)ae^{-a} - (-1)0e^{-0}\right] + (-1)e^{-x}\Big|_0^\infty \\
&= \lim_{a\to\infty} \left[-\frac{a}{e^a} + 0\right] + \lim_{b\to\infty} (-1)e^{-x}\Big|_0^b = -\lim_{a\to\infty} \frac{a}{e^a} + \lim_{b\to\infty} \left[(-1)e^b - (-1)e^0\right] \\
&= -\lim_{a\to\infty} \frac{a}{e^a} \begin{matrix}\to \infty \\ \to \infty\end{matrix} + \lim_{b\to\infty} \left[1 - \frac{1}{e^b}\right] = -\lim_{a\to\infty} \frac{\frac{d}{da}a}{\frac{d}{da}e^a} + [1 - 0] \\
&= -\lim_{a\to\infty} \frac{1}{e^a} + 1 = -0 + 1 = 1 \qquad \Box
\end{aligned}$$

b. We first compute $E(X)$, taking ashortcut or two over the approach used above to avoid dealing with limits, especially the idea that exponentials grow much faster than polynomials on the way to infinity.

$$E(X) = \int_{-\infty}^{\infty} xh(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_0^\infty x \cdot xe^{-x}\,dx = 0 + \int_0^\infty x^2 e^{-x}\,dx$$

Use parts with $u = x^2$ and $v = e^{-x}$, so $u' = 2x$ and $v = (-1)e^{-x}$.

$$\begin{aligned}
&= x^2(-1)e^{-x}\Big|_0^\infty - \int_0^\infty 2x(-1)e^{-x}\,dx = (-1)\frac{x^2}{e^x}\Big|_0^\infty + 2\int_0^\infty xe^{-x}\,dx \\
&= (-1)\frac{\infty^2}{e^\infty} - (-1)\frac{0^2}{e^0} + 2 \cdot 1 \quad \text{[By the work for } \textbf{a}\ \textit{iii}.] \\
&= (-1) \cdot 0 - 0 + 2 = 2 \quad \text{[Since } e^x \text{ grows much faster than } x^2 \text{ as } x \to \infty.]
\end{aligned}$$

We compute $E\left(X^2\right)$ in a similar way:

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 h(x)\,dx = \int_{-\infty}^{0} x^2 \cdot 0\,dx + \int_{0}^{\infty} x^2 \cdot xe^{-x}\,dx = 0 + \int_{0}^{\infty} x^3 e^{-x}\,dx$$

Use parts with $u = x^3$ and $v = e^{-x}$, so $u' = 3x^2$ and $v = (-1)e^{-x}$.

$$= x^3(-1)e^{-x}\Big|_0^\infty - \int_0^\infty 3x^2(-1)e^{-x}\,dx = (-1)\frac{x^3}{e^x}\Big|_0^\infty + 3\int_0^\infty x^2 e^{-x}\,dx$$

$$= (-1)\frac{\infty^3}{e^\infty} - (-1)\frac{0^3}{e^0} + 3 \cdot 2 \quad \text{[By the work for } E(X) \text{ above.]}$$

$$= (-1)\cdot 0 - 0 + 6 = 6 \quad \text{[Since } e^x \text{ grows much faster than } x^3 \text{ as } x \to \infty.\text{]}$$

It follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = 6 - 2^2 = 6 - 4 = 2$. ■

[Total = 8 × 15 = 120]

Part ♣. Bonus!

○●. There are 64 teams who play in a single elimination tournament (hence 6 rounds), and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points. *[1]*

SOLUTION. The expected number of points is $\frac{1}{2} \cdot 63 = 31.5$.

Why? Suppose G is agame in round s. Then you get 2^{s-1} points if you correctly predict the winner of G. The probability that random coin flips will correctly predict the winner in G and in the $s-1$ previous games involving the winner of G [otherwise the team in question doesn't get to round s] is $\frac{1}{2}\left(\frac{1}{2}\right)^{s-1} = \left(\frac{1}{2}\right)^s$. Thus the expected number of points for game G is $2^{s-1}\left(\frac{1}{2}\right)^s = \frac{1}{2}$. This is true for each and every game and there are 63 games. Thus the expected number of points obtained by the fair coin flipping strategy is $63 \cdot \frac{1}{2} = 31.5$. ■

●○. Write an original little poem about probability or mathematics in general. *[1]*

SOLUTION. Here is an original haiku by your instructor:

Your chance of passing
this probability test
improves with study.

Hey! No one said it had to be a good poem ... ■