Two Inequalities

(Partly from Ch. 8.)

or, what you might do if you know very little about your random variable.

Markov's Inequality

Suppose you have a random variable X s.t. $X \geq 0$ and $E(X)$ is defined. Then: For any $a>0$, $P(X \geq a) \leq \frac{E(X)}{a}$.

Example: Suppose X has an exponential density with $\lambda = 1$, ie the density is

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Then $E(X) = 1$. Suppose $a>0$.

$$P(X \geq a) = \int_a^\infty f(x)\,dx = \int_a^\infty e^{-x}\,dx$$

$u = -x$
$du = -dx$
$(-1)du = dx$

x	u
a	$-a$
∞	$-\infty$

$$= \int_{-a}^{-\infty} e^{u}(-1)\,du = (-1)e^{u}\Big|_{-a}^{-\infty}$$

$$= (-1)\underbrace{e^{-\infty}}_{= 0} + (+1)e^{-a} = e^{-a} = \frac{1}{e^{a}}$$

So $P(X \geq 2) = \frac{1}{e^2} \approx 0.1353$

Markov's Inequality tells us that

$$P(X \geq 2) \leq \frac{E(X)}{2} = \frac{1}{2} = 0.5$$

Note: Markov's Inequality is easy to use, but usually gives very crude estimates.

Chebyshev's Inequality

If X is a random variable with $E(X)$ and $V(X)$ both defined, then for all $k > 0$,

$$P(|X - E(X)| \geq k) \leq \frac{V(X)}{k^2}.$$

If $\mu = E(X)$ and $\sigma = \sqrt{V(X)}$, then this looks like

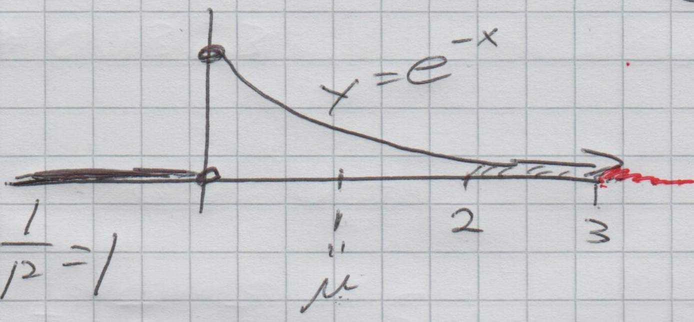
$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Example: If X has an exponential density with $\lambda = 1$, $\mu = E(X) = 1$ & $\sigma^2 = V(X) = 1$.

$$P(X \geq 2) = ?$$

$$\parallel$$

$$P(\underbrace{|X-1| \geq 1}_{k}) \leq \frac{V(X)}{k^2} = \frac{1}{1^2} = 1$$



So this was pretty useless...

With $a = 3$?

Markov's Inequality

$$P(X \geq a) = P(X \geq 3) \leq \frac{E(X)}{a} = \frac{1}{3}$$

$$\parallel$$

$$\frac{1}{e^a}$$

Chebyshev's Inequality

$$\parallel$$

$$P(|X-1| \geq 2) \leq \frac{V(X)}{2^2} = \frac{1}{4}$$

$$\to \frac{1}{e^3} \approx 0{,}0498$$

These inequalities usually give only crude upper bounds, but that's better than nothing.