Sums of Independent Random Variables

or, the confusing "convolutions" (Chapter 7)

Suppose X and Y are discrete independent random variables, with probability distribution functions m_X and m_Y, respectively. Define a new random variable by $Z = X + Y$.

Q: What is the probability distribution function m_Z of Z?

$$m_Z(a) = P(Z=a) = \sum_{k=-\infty}^{\infty} P(X=k)\,P(Y=a-k)$$

$$(m_X * m_Y)(a) = \underbrace{\sum_{k=-\infty}^{\infty} m_X(k)\,m_Y(a-k)}_{\text{"convolution sum"}}$$

The convolution of m_X and m_Y.

Example: X counts the number of heads that come up in a single toss of a fair coin.

Y —— " ——————— (different toss!)

$$m_X(k) = m_Y(k) = \begin{cases} \frac{1}{2} & \text{if the coin came up heads } k=1 \\ \frac{1}{2} & \text{if the coin came up tails } k=0 \\ 0 & k \neq 0,1 \end{cases}$$

Let $Z = X+Y$ [counts the number of heads in two tosses of a fair coin]

Z could be 0, 1, or 2

$$m_Z(0) = (m_X * m_Y)(0) = \sum_{k=-\infty}^{\infty} m_X(k) \cdot m_Y(0-k)$$

Since $m_X(k) = 0$ unless $k = 0$ or 1

$$= \sum_{k=0}^{1} m_X(k)\, m_Y(0-k)$$

$$= m_X(0)\, m_Y(0-0) + m_X(1)\, m_Y(0-1)$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 0 = \frac{1}{4}$$

$m_Z(0) = P(TT) = \frac{1}{4}$

$$m_Z(1) = (m_X * m_Y)(1)$$

$$= \sum_{k=-\infty}^{\infty} m_X(k)\, m_Y(1-k)$$

Since $m_X(k) = 0$ unless $k = 0$ or 1,

$$= m_X(0)\, m_Y(1-0) + m_X(1)\, m_Y(1-1)$$

$$= m_X(0)\, m_Y(1) + m_X(1)\, m_Y(0)$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

→ P(exactly 1 head in two tosses) $= P(TH) + P(HT)$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$m_Z(2) = (m_X * m_Y)(2) = \sum_{k=-\infty}^{\infty} m_X(k)\, m_Y(2-k)$$

Since $m_X(k) = 0$ unless $k = 0$ or 1

$$= m_X(0)\, m_Y(2-0) + m_X(1)\, m_Y(2-1)$$

$$= m_X(0)\, m_Y(2) + m_X(1)\, m_Y(1)$$

$$= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

→ P(2 heads in two tosses) $= P(HH) = \frac{1}{4}$.

Suppose X and Y are independent continuous random variables, with probability density functions f_X and f_Y, respectively. Let $Z = X+Y$.

Q.: What is the density function f_Z of Z?

$$f_Z(a) = (f_X * f_Y)(a) = \int_{-\infty}^{\infty} f_X(t)\, f_Y(a-t)\, dt$$

"convolution integral"

convolution of f_X and f_Y

Example: X and Y both have a uniform density on $[0,1]$

ie $f_X(a) = f_Y(a) = \begin{cases} 1 & \text{if } 0 \le a \le 1 \\ 0 & \text{otherwise} \end{cases}$

What is $f_Z(a) = (f_X * f_Y)(a)$

$$= \int_{-\infty}^{\infty} f_X(t)\, f_Y(a-t)\, dt$$

$$\int_{-\infty}^{\infty} f_X(t)\, f_Y(a-t)\,dt$$

Note that $f_X(t) = 0$ if $x \notin [0,1]$, and $=1$ if $x \in [0,1]$,

$$= \cancel{\int_{-\infty}^{0} 0 \cdot f_Y(a-t)\,dt} + \int_0^1 1 \cdot f_Y(a-t)\,dt + \cancel{\int_1^{\infty} 0 \cdot f_Y(a-t)\,dt}$$

$$= \int_0^1 f_Y(a-t)\,dt$$

If $a-t \in [0,1]$, then $f_Y(a-t) = 1$, otherwise $f_Y(a-t) = 0$.

$$\cancel{= \int_0^{a-1} 0 \cdot dt + \int_{a-1}^{a} 1\,dt}$$

$0 \le a-t \le 1$ $\quad (\& \; 0 \le t \le 1)$

$\Rightarrow 0 \ge t-a \ge -1$

$a \ge t \ge a-1$

$=$ if $0 \le a \le 1$, then

$$= \int_0^a 1\,dt + \cancel{\int_a^1 0\,dt} = t\Big|_0^a = a - 0 = a$$

if $1 \le a \le 2$

$$= \cancel{\int_0^{a-1} 0\,dt} + \int_{a-1}^1 1\,dt = t\Big|_{a-1}^1 = 1-(a-1) = 2-a$$

if $a \notin [0,2]$

then $f_Z(a) = 0$.

So the density of Z is

$$f_Z(a) = (f_X * f_Y)(a) = \begin{cases} a & \text{if } 0 \le a \le 1 \\ 2-a & \text{if } 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The convolution of two normal distributions is another normal distributions.

Try "convolving" two standard normal densities:

$$g(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$