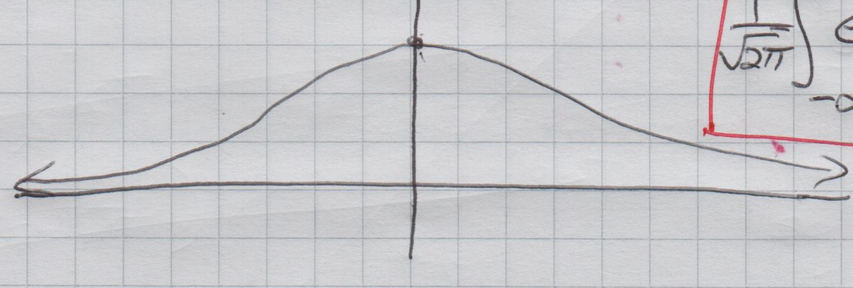
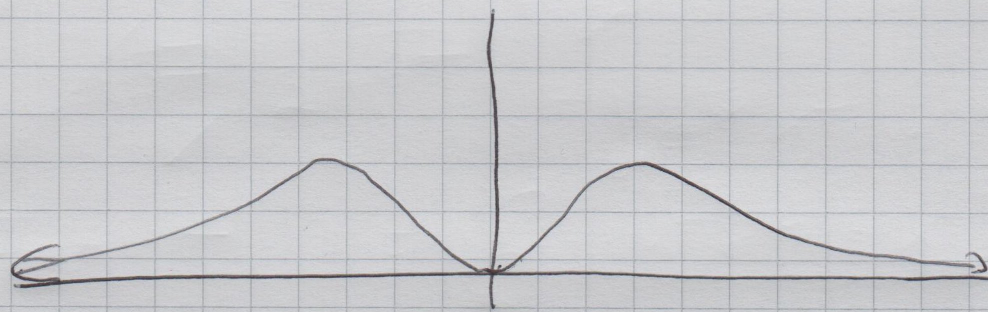
Example: A pretty hard continuous density function

(Standard)
Normal density: $\varphi(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

$$\boxed{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1}$$



Our density: $f(x) = \dfrac{1}{\sqrt{2\pi}} \cdot x^2 e^{-x^2/2}$



Check this is a valid density:

1) $f(x) \geq 0$ since $\frac{1}{\sqrt{2\pi}} > 0$, $x^2 \geq 0$, and $e^{-x^2/2} > 0$

2) $f(x)$ is continuous for all x because x^2 and $e^{-x^2/2}$ are continuous for all x
(Hence $f(x)$ is integrable.)

3) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx$

$$\left[\text{Since } f(x) = f(-x) \text{ for all } x, \ (f(x) \text{ is } \underline{\text{even}}) \quad \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx \right]$$

$$= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx = 2\lim_{a\to\infty} \int_{0}^{a} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{0}^{a} x^2 e^{-x^2/2}\,dx$$

$\cancel{u = x^2 \quad v' = e^{-x^2/2}}$

$\cancel{u' = 2x \quad v = ?}$

$u = x \quad v' = x e^{-x^2/2}$

$u' = 1$

$v = \int x e^{-x^2/2}\,dx \qquad w = -\frac{x^2}{2}, \quad dw = -\frac{2x}{2}\,dx, \quad x\,dx = (-1)\,dw$

$= \int e^{w} (-1)\,dw$

$= -e^{w} = (-1)e^{-x^2/2}$

$$= \frac{2}{\sqrt{2\pi}} \lim_{a\to\infty} \left[(-1)x e^{-x^2/2} \Big|_{0}^{a} + \int_{0}^{a} 1\cdot(+1) e^{-x^2/2}\,dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\lim_{a\to\infty} \left[(-1)a e^{-a^2/2} - \cancel{(-1)\cdot 0\, e^{-0^2/2}} \right] + \lim_{a\to\infty} \int_{0}^{a} e^{-x^2/2}\,dx \right]$$

$$= \left[\lim_{a\to\infty} \frac{2}{\sqrt{2\pi}} (-1)\underset{\infty}{a}\, \underset{\to 0}{e^{-a^2/2}} \right] = 0$$

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$$+ \left[\lim_{a\to\infty} \frac{2}{\sqrt{2\pi}} \int_{0}^{a} e^{-x^2/2}\,dx \right]$$

$$= 0 + \lim_{a \to \infty} \frac{2}{\sqrt{2\pi}} \int_0^a e^{-x^2/2}\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2}\,dx = 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} \varphi(x)\,dx = 1$$

So $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx = 1,$

as required.

What is the expected value of a random variable X that has f(x) as its density?

$$E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \overset{x^2 \cdot x}{x^3} e^{-x^2/2}\,dx$$

Try $w = -\frac{x^2}{2}$, so

$dw = \frac{-2x}{2}\,dx = (-1)x\,dx$

$(-1)\,dw = x\,dx$

Notice that $x^2 = -2w$.

$$= \int_{x=-\infty}^{x=\infty} \frac{1}{\sqrt{2\pi}} (-2w) e^{w} (-1)\,dw$$

$$= \int_{x=-\infty}^{x=\infty} \frac{2}{\sqrt{2\pi}} w e^{w}\,dw$$

$u = w \qquad v' = e^w$

$u' = 1 \qquad v = e^w$

$$= \frac{2}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} w e^{w} dw = \frac{2}{\sqrt{2\pi}} w e^{w} \Big|_{x=-\infty}^{x=\infty} - \frac{2}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{w} dw$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(w e^{w} - e^{w} \right) \Big|_{x=-\infty}^{x=\infty} \right.$$

$$= \frac{2}{\sqrt{2\pi}} \left(-\frac{x^2}{2} e^{-x^2/2} - e^{-x^2/2} \right) \Big|_{-\infty}^{\infty}$$

Here we get away with tossing $\pm\infty$ about like a number, because we're dealing products of powers of x & exponentials.

$$= \frac{2}{\sqrt{2\pi}} \left(-\underbrace{\overbrace{\frac{\infty^2}{2}}^{\to\infty} \overbrace{e^{-\infty^2/2}}^{\to 0}}_{\to 0} - \overbrace{e^{-\infty^2/2}}^{\to 0} \right)$$

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$$- \frac{2}{\sqrt{2\pi}} \left(-\underbrace{\overbrace{\frac{(-\infty)^2}{2}}^{\to\infty} \overbrace{e^{-(-\infty)^2/2}}^{\to 0}}_{\to 0} - \overbrace{e^{-(-\infty)^2/2}}^{\to 0} \right)$$

$$= 0$$

For practice, compute $V(X)$.

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2} dx = \ldots$$