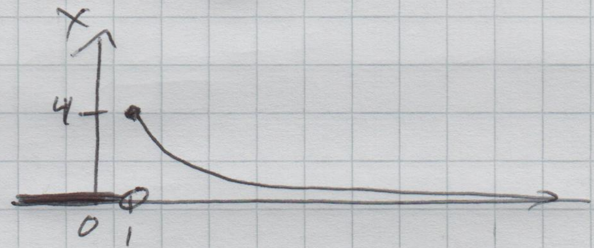
Example: A pretty easy continuous density function or two

Suppose that the continuous random variable X has the density function

$$g(x) = \begin{cases} \frac{4}{x^5} & x \geq 1 \\ 0 & x < 1 \end{cases}$$



Is $g(x)$ a valid probability density function?

1) $g(x)$ is integrable because it has only one jump discontinuity

2) $\frac{4}{x^5} \geq 0$ if $x \geq 1$ and $0 \geq 0$ (if $x < 1$)

so $g(x) \geq 0$ for all x

3) Is $\int_{-\infty}^{\infty} g(x)\,dx = 1$?

$$\int_{-\infty}^{\infty} g(x)\,dx = \cancel{\int_{-\infty}^{1} 0\,dx}^{=0} + \int_{1}^{\infty} \frac{4}{x^5}\,dx$$

$$= \int_{1}^{\infty} 4x^{-5}\,dx = \lim_{a\to\infty} \int_{1}^{a} 4x^{-5}\,dx$$

$$= \lim_{a\to\infty} \left(4 \cdot \frac{x^{-5+1}}{-5+1} \Big|_{1}^{a} \right)$$

$$= \lim_{a\to\infty} \left(\cancel{4} \cdot \frac{x^{-4}}{-\cancel{4}} \right) \Big|_{1}^{a} = \lim_{a\to\infty} \left(\frac{-1}{x^4} \Big|_{1}^{a} \right)$$

$$= \lim_{a\to\infty} \left(\left(\frac{-1}{a}\right) - \left(\frac{-1}{1}\right) \right) = \lim_{a\to\infty} \left(\underbrace{\frac{-1}{a}}_{\to 0} + 1 \right)$$

$$= 0 + 1 = 1 \checkmark$$

So $g(x)$ is a valid probability density function.

es $P(2 \leq X \leq 3) = \int_{2}^{3} g(x)\,dx = \int_{2}^{3} \frac{4}{x^5}\,dx = \frac{-1}{x^4}\Big|_{2}^{3}$

$$= \left(\frac{-1}{3^4}\right) - \left(\frac{-1}{2^4}\right) = \frac{-1}{81} + \frac{1}{16} = \frac{1}{16} - \frac{1}{81}$$

$$E(X) = \int_{-\infty}^{\infty} x g(x)\,dx = \cancel{\int_{-\infty}^{1} x\cdot \underbrace{0}_{=0}\,dx} + \int_{1}^{\infty} x\cdot \frac{4}{x^5}\,dx$$

$$= \int_{1}^{\infty} \frac{4}{x^4}\,dx = \int_{1}^{\infty} 4x^{-4}\,dx = \lim_{a\to\infty} \int_{1}^{a} 4x^{-4}\,dx$$

Power Rule again.

$$= \lim_{a\to\infty} \left(4\cdot \frac{x^{-4+1}}{-4+1} \Big|_{1}^{a} \right) = \lim_{a\to\infty} \left(4\cdot \frac{x^{-3}}{-3} \Big|_{1}^{a} \right)$$

$$= \lim_{a\to\infty} \left(\frac{-4}{3}\cdot x^{-3} \right) \Big|_{1}^{a} = \lim_{a\to\infty} \left(\frac{-4}{3}\cdot \frac{1}{x^3} \Big|_{1}^{a} \right)$$

$$= \lim_{a\to\infty} \left(\left(-\frac{4}{3}\cdot\frac{1}{a^3}\right) - \left(-\frac{4}{3}\cdot\frac{1}{1^3}\right) \right)$$

$$= \lim_{a\to\infty} \frac{4}{3}\left(1 - \underbrace{\frac{1}{a^3}}_{0}\right) = \frac{4}{3}(1-0) = \frac{4}{3}$$

So the expected value of the random variable X with density $g(x)$ is $E(X) = \frac{4}{3}$.

What about the variance?

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \; ? \; - \left[\frac{4}{3}\right]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 g(x)\,dx = \overset{=0}{\cancel{\int_{-\infty}^{1} x^2 \cdot 0\,dx}} + \int_{1}^{\infty} x^2 \cdot \frac{4}{x^5}\,dx$$

$$= \int_{1}^{\infty} \frac{4}{x^3}\,dx = \int_{1}^{\infty} 4x^{-3}\,dx = \lim_{a\to\infty} \int_{1}^{a} 4x^{-3}\,dx$$

$$= \lim_{a\to\infty} \left(4 \cdot \frac{x^{-3+1}}{-3+1} \Big|_1^a \right) = \lim_{a\to\infty} \left(4 \cdot \frac{x^{-2}}{-2} \Big|_1^a \right)$$

$$= \lim_{a\to\infty} \left(-\frac{2}{x^2} \Big|_1^a \right) = \lim_{a\to\infty} \left(\left(-\frac{2}{a^2}\right) - \left(-\frac{2}{1^2}\right) \right)$$

$$= \lim_{a\to\infty} \left(2 - \underbrace{\frac{2}{a^2}}_{\to 0} \right) = 2 - 0 = 2$$

Thus $V(X) = E(X^2) - [E(X)]^2$

$$= 2 - \left[\frac{4}{3}\right]^2 = 2 - \frac{16}{9}$$

$$= \frac{18}{9} - \frac{16}{9} = \frac{2}{9}.$$

A little more generally if the continuous random variable X has

density $$h(x) = \begin{cases} (n-1)x^{-n} = \dfrac{n-1}{x^n} & x \geq 1 \\ 0 & x < 0 \end{cases}$$

then (a) this is a valid density (if $n \geq 2$)

(b) $E(X) = \dfrac{n-1}{n-2}$ (if $n \geq 3$)

(c) $V(X) = \dfrac{n-1}{n-3} - \left(\dfrac{n-1}{n-2}\right)^2$ (if $n \geq 4$).

Practice by working it out for yourself!