Variance and Standard Deviation

(Section 6.2 & part of 6.3)

If X is a numerically-valued random variable, then the expected value of X is

$$E(X) = \sum_{x \in \Omega} x\, m(x) \quad \text{, if } X \text{ is discrete w. prob. fn. } m$$

$$= \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{if } X \text{ is cts. w. density fn. } f.$$

Recall that if X counts the number of heads in a single toss of a fair coin then $E(X) = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$

... which is a value X can never have.

Q.: How far is $E(X)$ likely to be from X?

~~Naive:~~ $E(X - E(X)) = E(X) - E(\overset{\text{constant}}{E(X)})$

$= E(X) - E(X) = 0$

Possible try: $E(|X-E(x)|)$?

Hard to handle ...

What people really do is look at:

<u>Variance of X</u> $= V(X)$

$$= E\left(\left[X-E(x)\right]^2\right)$$

$$= E\left(X^2 - XE(X) - XE(X) + (E(x))^2\right)$$

$$= E(X^2) - E(XE(X)) - E(XE(X)) + (E(x))^2$$

$$= E(X^2) - 2(E(x))^2 + (E(X))^2$$

$$= E(X^2) - (E(x))^2$$

<u>Standard deviation of X</u>

sigma

$$\sigma = \sqrt{V(X)} = \sqrt{E(X^2)-(E(X))^2}$$

$$= \sqrt{E((X-E(X))^2)}$$

eg X counts heads in a single toss of a fair coin

$E(X) = \frac{1}{2}$

$E(X^2) = 1^2 \cdot \frac{1}{2} + 0^2 \cdot \frac{1}{2}$

$= \frac{1}{2}$

$V(X) = E(X^2) - (E(X))^2$

$= \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\sigma = \sqrt{V(X)} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Whichever outcome, 0 or 1, you get you're 1σ away from $E(X)$.

eg Uniform distribution on $[-1, 3]$

density function

$$f(x) = \begin{cases} \frac{1}{4} & -1 \le x \le 3 \\ 0 & x < -1 \text{ or } x > 3 \end{cases}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X) = \frac{3+(-1)}{2} = \frac{2}{2} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \cancel{\int_{-\infty}^{-1} x^2 \cdot 0\, dx} + \int_{-1}^{3} x^2 \cdot \frac{1}{4}\, dx + \cancel{\int_{3}^{\infty} x^2 \cdot 0\, dx}$$

$$= \frac{1}{4} \cdot \frac{x^3}{3} \Big|_{-1}^{3} = \frac{3^3}{12} - \frac{(-1)^3}{12}$$

$$= \frac{27}{12} - \frac{-1}{12}$$

$$= \frac{28}{12} = \frac{7}{3}$$

$$V(X) = E(X^2) - [E(X)] = \frac{7}{3} - (1)^2 = \frac{7}{3} - \frac{3}{3} = \frac{4}{3} = \frac{(3-(-1))^2}{12}$$

Uniform cts. distribution on $[a, b]$

$$E(X) = \frac{a+b}{2} \qquad V(X) = \frac{(b-a)^2}{12}$$

eg Suppose X has a standard normal distribution

i.e. density function $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$u = -x^2/2$

$du = (-1) x\, dx$

x	u
0	0
a	$-a^2/2$
b	$-b^2/2$

$$= \int_0^{\infty} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_{-\infty}^{0} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \lim_{a\to\infty} \int_0^{a} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx + \lim_{b\to-\infty} \int_b^{0} \frac{x}{\sqrt{2\pi}} e^{-x^2} dx$$

$\int -e^{-u} du = e^{-u} = e^{-x^2/2}$

$$= \lim_{a\to\infty} (-1) \int_0^{-a^2/2} \frac{1}{\sqrt{2\pi}} e^{u} du + \lim_{b\to-\infty} (-1) \int_{-b^2/2}^{0} \frac{1}{\sqrt{2\pi}} e^{u} du$$

$$= \lim_{a\to\infty} \frac{(-1)}{\sqrt{2\pi}} e^{u} \Big|_0^{-a^2/2} + \lim_{b\to-\infty} \frac{(-1)}{\sqrt{2\pi}} e^{u} \Big|_{-b^2/2}^{0}$$

$$= -\frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \left(\underbrace{e^{-a^2/2}}_{\to 0} - e^{0}\right) - \frac{1}{\sqrt{2\pi}} \lim_{b\to-\infty} \left(e^{0} - \underbrace{e^{-b^2/2}}_{\to 0}\right)$$

$$= \frac{-1}{\sqrt{2\pi}}(-1) - \frac{1}{\sqrt{2\pi}} \cdot 1 = 0$$

So for standard normal, $\mu = E(X) = 0$.

$$E(X^2) = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx$$ and its even...

$$= 2\int_0^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Use integration by parts

$u = x^2 \quad \cancel{v' = e^{-x^2/2}}$

$\cancel{u' = 2x} \quad v = ?$

$u = e^{-x^2/2} \quad v' = x^2$

$u' = e^{-x^2/2}\frac{d}{dx}\left(\frac{-x^2}{2}\right) \quad v = \frac{x^3}{3}$

$= -\frac{2x}{2} e^{-x^2/2} = -x e^{-x^2/2}$

$$= \frac{2}{\sqrt{2\pi}}\left[uv - \int u'v\,dx\right]$$

$$\cancel{= \frac{2}{\sqrt{2\pi}}\left[\frac{x^3}{2} e^{-x^2/2}\Big|_0^{\infty} - \int_0^{\infty} -\frac{x^4}{2} e^{-x^2/2} dx\right]}$$

This time try

$u = x \quad v' = xe^{-x^2}$

$u' = 1 \quad v = -e^{-x^2/2}$

$$= \frac{2}{\sqrt{2\pi}}\left[-xe^{-x^2/2}\Big|_0^{\infty} - \int_0^{\infty} (-1) e^{-x^2/2} dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\cancel{-\infty \underbrace{e^{-\infty/2}}_{0} - (\underbrace{-0e^{-0/2}}_{0})} + \int_0^{\infty} e^{-x^2/2} dx\right]$$

~~exponentials beat powers of x~~

$$= 2\int_0^{\infty} \frac{1}{2\sqrt{\pi}} \cdot e^{-x^2/2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$$

'cause this a valid density.

so $V(X) = E(X^2) - [E(X)]^2 = 1 - 0^2 = 1$

& $\sigma = \sqrt{V(X)} = 1$.

Basic Properties of $V(X)$

1) $V(cX) = c^2 V(X)$ for any constant c

2) $V(X+Y) = V(X) + V(Y)$ if X & Y are independent

(could happen if they're not independent, but very unlikely)