Expected Value

(Section 6.1 and part of Section 6.3.)

Suppose X is a random variable which has numerical outcomes. The expected value of X is a weighted average of the values X can have, weighted by their probabilities.

$$E(X) = \sum_{x \in \Omega} x \cdot m(x)$$ if X is discrete (m is the probability distribution function)

$$= \int_{-\infty}^{\infty} x f(x)\, dx$$ if X is cts. (f is the probability density function)

eg Toss a coin (a fair one) once.

X = # heads that come up $\left[m(1) = \frac{1}{2} = m(0) \right]$

$$E(X) = 1 \cdot m(1) + 0 \cdot m(0) = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$$

Note that $E(X)$ could be a value X never has...

eg $\Omega = \{1, 2, 3, 4, 5\}$

with a uniform distribution $\left[m(\omega) = \frac{1}{5}\right]$

X is the unknown outcome

$$E(X) = 1\cdot\frac{1}{5} + 2\cdot\frac{1}{5} + 3\cdot\frac{1}{5} + 4\cdot\frac{1}{5} + 5\cdot\frac{1}{5}$$

$$= \frac{1+2+3+4+5}{5}$$ which is really the average of $1, 2, 3, 4, 5$.

eg

X ~~has~~ is the ~~d~~ density function

$$f(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & x < 1 \text{ or } x > 3 \end{cases}$$

(Uniform continuous distribution on $[1,3]$)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \cancel{\int_{-\infty}^{1} x\cdot 0\,dx} + \int_{1}^{3} x\cdot\frac{1}{2}\,dx + \cancel{\int_{3}^{\infty} x\cdot 0\,dx}$$

$$= \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^3 = \frac{x^2}{4}\Big|_1^3$$

$$= \frac{3^2}{4} - \frac{1^2}{4} = \frac{9}{4} - \frac{1}{4} = \frac{8}{4} = 2$$

A uniform cts. distribution on $[a,b]$ has expected value $E(X) = \frac{b+a}{2}$.

Basic properties of E(X)

1) $E(X+Y) = E(X) + E(Y)$ — X, Y are both discrete or both cts. random variables.

2) $E(cX) = cE(X)$ — for any constant c.

3) If X & Y are independent random variables, (ie Events for X are always independent of events for Y) then $E(XY) = E(X)E(Y)$.

[This could happen if X&Y are not independent, but that's not the way to bet...]

4) If $g(x)$ is a function,

$$E(g(X)) = \sum_{x\in\Omega} g(x)\,m(x) \quad \text{if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} g(x)\,f(x)\,dx \quad \text{if } X \text{ is cts.}$$

Rarely do you get $E(g(X)) = g(E(X))$ unless $g(x)$ is a linear function.

A possible pitfall is that $E(X)$ is actually undefined for some random variable.

<u>eg</u> Suppose X is a continuous random variable with density $f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1. \end{cases}$

(We checked previously that this is a valid density function.)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \cancel{\int_{-\infty}^{1} x \cdot 0\,dx} + \int_{1}^{\infty} x \cdot \frac{1}{x^2}\,dx$$

$$= \int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x}\,dx$$

$$= \lim_{a \to \infty} \ln(x) \Big|_{1}^{a}$$

$$= \lim_{a \to \infty} \left[\ln(a) - \cancel{\ln(1)}\right] = \lim_{a \to \infty} \ln(a)$$

$$= \infty$$

so the limit is not defined, and hence $E(X)$ is undefined.