Conditional Probability (Ch. 4 in the textbook)

- We'll cover only the basics

Suppose A and B are some events.
Then the conditional probability of A given B is $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

Two events A and B are independent if $P(A|B) = P(A)$,

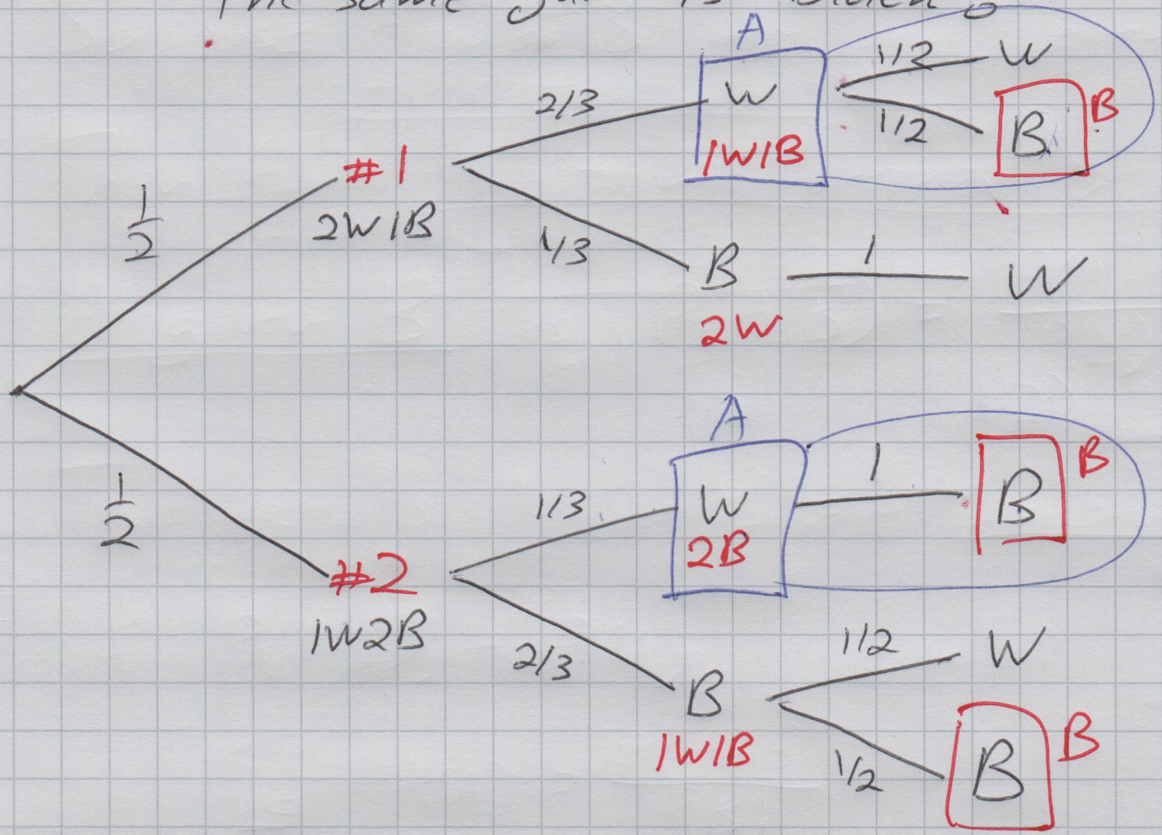
ie $P(A) = \frac{P(A \cap B)}{P(B)}$

$\Longleftrightarrow P(A \cap B) = P(A)P(B)$.

Example: You have 6 marbles, 3 white and 3 black, with 2 white and 1 black in a jar (#1), and 2 black and 1 white in another identical jar (#2).

You pick a jar at random and then randomly pick a marble out of that jar.

Q: If we picked out a white ~~marble~~ marble, ~~what~~ what is the probability that the next marble we choose from the same jar is black?



Events: A: We ~~picked~~ got a white marble on the selection of the first marble.

B: We got a black marble on the selection of the second marble.

We're asking for $P(B|A) = \frac{P(B \cap A)}{P(A)}$.

$P(A) = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{2}\cdot\frac{3}{3} = \frac{1}{2}$

$P(B) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot 1 + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2} = \frac{3}{6} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot 1 = \frac{2}{6} = \frac{1}{3}$

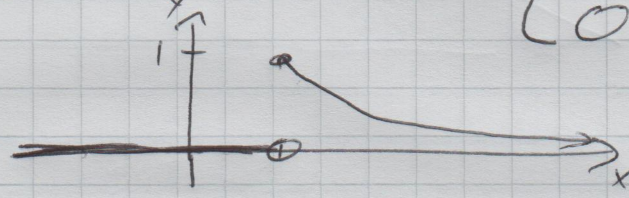
So $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$

[$\neq \frac{1}{2} = P(B)$

so A & B are not independent events]

[By the way, if A is independent of B, then B is also independent of A.]

<u>Example:</u> Suppose X is a continuous random variable with density function $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$



<u>Aside:</u> This a valid density function because

(1) $f(x) \geq 0$ (since $0 \geq 0$ and $\frac{1}{x^2} \geq 0$ when $x \geq 1$)

(2) $f(x)$ is integrable because it's continuous except at $x=1$, where it has a jump discontinuity

(3) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} x^{-2}\,dx = 0 + \frac{x^{-2+1}}{-2+1}\Big|_1^{\infty}$

$= \frac{-1}{x}\Big|_1^{\infty} = \left(\frac{-1}{\infty}\right) - \left(\frac{-1}{1}\right) = -0 + 1 = 1$ ✓

Let $A = [-1, 3]$

& $B = [2, 4]$, so $A \cap B = [2, 3]$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/4} = \frac{4}{6} = \frac{2}{3}$$

$$P(B) = \int_2^4 f(x)\,dx = \int_2^4 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_2^4 = \frac{-1}{4} - \left(\frac{-1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$P(A \cap B) = \int_2^3 f(x)\,dx = \int_2^3 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_2^3 = \frac{-1}{3} - \left(\frac{-1}{2}\right)$$

$$= -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$

$$P(A) = \int_{-1}^3 f(x)\,dx = \int_{-1}^1 0\,dx + \int_1^3 \frac{1}{x^2}\,dx = 0 + \frac{-1}{x}\Big|_1^3$$

$$= -\frac{1}{3} - \left(\frac{-1}{1}\right) = -\frac{1}{3} + 1 = \frac{2}{3}$$

Since $P(A) = P(A|B)$, A & B are independent.

Bayes' Formula

Suppose you have two events A & B such that $A = \Omega \setminus B$

(so $A \cap B = \emptyset$ and $A \cup B = \Omega$).

If E is any other event,

$$P(A|E) = \frac{P(A)P(E|A)}{P(A)P(E|A) + P(B)P(E|B)}.$$