Continuous Probability: The Basics

Needed when we have an experiment that results in a measurement that is some real number.

Sample space: $\mathbb{R}$

Events are subsets (usually intervals) of $\mathbb{R}$.

The probability of an event is given by integrating:

$$P(A) = \int_A f(x)\,dx$$

where $f(x)$ is a probability density function.

Such a function must satisfy

$$0 \leq P(a \leq X \leq b) = \int_a^b f(x)\,dx \leq 1$$

random variable representing the measurement

So we usually require that a probability density function must satisfy the following requirements:

(1) $f(x)$ has to be defined and integrable on $\mathbb{R}$.

(2) $f(x) \geq 0$ for all $x \in \mathbb{R}$

(3) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ "Improper Integral"

$$\text{"} = \text{"} \lim_{a \to -\infty} \int_a^0 f(x)\,dx + \lim_{b \to \infty} \int_0^b f(x)\,dx$$

If X is a random variable that is continuous, and $f(x)$ is its density function, then the cumulative distribution function is

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = P(X \leq x).$$

Example: We choose a number at random from the interval $[0,1]$.

A ~~n~~ density function that keeps the outcomes as fair as possible is

$$f(x) = \begin{cases} 0 & \text{if } x<0 \text{ or if } x>1 \\ 1 & \text{if } 0\leq x\leq 1 \end{cases}$$

This is a valid density function:

(1) $f(x)$ is defined on all of $\mathbb{R}$ and is integrable because it's ~~integrable~~ continuous everywhere except at 0 & 1, where it has jump discontinuities ✓

(2) $f(x) \geq 0$ for all x (since $0\geq 0$ & $1\geq 0$) ✓

(3) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 1\,dx + \int_{1}^{\infty} 0\,dx$$

$$= 0 + x\Big|_0^1 + 0$$

$$= 1-0 = 1 \quad ✓$$

Q: What is the probability that our random number is $< \frac{1}{4}$?

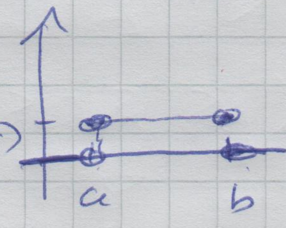
$$P(X<\tfrac{1}{4}) = \int_{-\infty}^{1/4} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/4} 1\,dx$$

$$= 0 + x\Big|_0^{1/4} = \frac{1}{4} - 0 = \frac{1}{4} = 0.25$$

Uniform Continuous Distribution on $[a,b]$

Has density function

$$f(x) = \begin{cases} 0 & x<a \\ \dfrac{1}{b-a} & a \le x \le b \\ 0 & x>b \end{cases}$$



$$1 = A = l \cdot w = (b-a)\cdot\frac{1}{b-a}$$

Example: We have a radioactive sample and a Geiger counter, set up so that at any given instant the probability of the next particle being detected within 1 second is $\frac{1}{2}$.

Q.: What is the probability that we will detect the next particle after 2 seconds?

The appropriated density function is

$$g(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2} e^{-x/2} & x > 0 \end{cases}$$

Check this a valid density function:

(1) defined for all x and continuous except for a jump discontinuity at 0, so it's integrable

(2) $g(x) \ge 0$ when $x \le 0$ and when $x > 0$ (since $e^t > 0$ for all t)

(3) $\int_{-\infty}^{\infty} g(x)\,dx = \cancel{\int_{-\infty}^{0} 0\,dx} + \int_0^{\infty} \frac{1}{2} e^{-x/2}\,dx$

Substitute $u = -\frac{x}{2}$

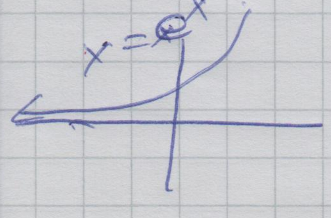
$du = -\frac{1}{2}\,dx$

$(-1)du = \frac{1}{2}\,dx$

x	u
0	0
∞	$-\infty$

$= \int_0^{-\infty} (-1) e^u\,du = \int_{-\infty}^{0} e^u\,du$

$= e^u \Big|_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1$



Q.: What is the probability that we will detect the next particle after 2 seconds?

$$P(X>2) = \int_2^\infty g(x)\,dx = \int_2^\infty \frac{1}{2}e^{-x/2}\,dx$$

$$= 1 - P(X \le 2)$$

$$= 1 - \int_{-\infty}^{2} g(x)\,dx = 1 - \left(\cancel{\int_{-\infty}^{0} 0\,dx} + \int_0^2 \frac{1}{2}e^{-x/2}\,dx\right)$$

$u = \frac{-x}{2}$

x	u
0	0
2	-1

$$= 1 - \left((-1)\int_0^{-1} e^u\,du\right)$$

$$= 1 - \int_{-1}^{0} e^u\,du = 1 - \left(e^u\Big|_{-1}^{0}\right)$$

$$= 1 - (e^0 - e^{-1}) = \cancel{1} - (\cancel{1} - \tfrac{1}{e}) = \frac{1}{e}$$

Example of an <u>Exponential Distribution</u>

Has density function $g(x) = \begin{cases} 0 & x \le 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$

(Where $\lambda > 0$ is a constant.)

In our example, we have $\lambda = \frac{1}{2}$. (§2.2)