Discrete Probability continued some more:

Some Uses of Permutations and Combinations

Hat Check Problem

n people check their hats at the opera and them back randomly

Q.: What is the probability that at least one person gets their own hat back?

The assignment of hats to people when returned is a permutation of $1, \ldots, n$.

A_i = the event that person i gets their own hat back

$$P(A_i) = \frac{\#\text{permutations of } 1,\ldots,n \text{ that leave } i \text{ fixed}}{\#\text{permutations of } 1,\ldots,n}$$

$$= \frac{(n-1)!}{n!} = \frac{\cancel{(n-1)(n-2)\cdots(2)(1)}}{n\cancel{(n-1)(n-2)\cdots(2)(1)}} = \frac{1}{n}$$

$$P(A_i \cap A_j) = \frac{\#\text{permutations of } 1,\ldots,n \text{ leave } i \,\&\, j \text{ fixed}}{\#\text{permutations of } 1,\ldots,n}$$

$$= \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$P(A_i \cap A_j \cap A_k) = \frac{\#\text{permutations of } 1,\ldots,n \text{ leaving } i,j,k \text{ fixed}}{\#\text{permutations of } 1,\ldots,n}$$

$$= \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)} \cdots$$

$\vdots$

P(at least one person gets their own hat back)

$= P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$

$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - \cdots - P(A_{n-1} \cap A_n)$

$+ P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + \cdots + P(A_{n-2} \cap A_{n-1} \cap A_n)$

$- P(A_1 \cap A_2 \cap A_3 \cap A_4) - \cdots - P(A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n)$

$+ \cdots$ "Inclusion-Exclusion"

$\vdots$ Generalization of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$+ (-1)^{n+1} P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n)$

$= \frac{1}{n} + \cdots + \frac{1}{n} \qquad = n\frac{1}{n} \qquad = 1$

$- \binom{n}{2} \frac{1}{n(n-1)} \qquad = -\frac{n(n-1)}{2!} \cdot \frac{1}{n(n-1)} \qquad -\frac{1}{2!}$

$+ \binom{n}{3} \frac{(n-3)!}{n!} \qquad + \frac{n!}{(n-3)!3!} \cdot \frac{(n-3)!}{n!} \qquad + \frac{1}{3!}$

$\vdots \qquad\qquad -\frac{1}{4!}$

$+ (-1)^{n+1} \binom{n}{n} \frac{n!}{n!} \qquad \vdots$

$+ (-1)^{n+1} \frac{1}{n!}$

$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n+1} \frac{1}{n!}$

Binomial distribution

Experiment: do n Bernoulli trials (with probability p of success and $q = 1-p$ of failure on each trial).

Q.: What is the probability of getting exactly k successes in those n trials?

$$P(k \text{ successes}) = \binom{n}{k} p^k q^{n-k}$$

ways to select which k of the n outcomes were successes

Example: Toss a fair coin 7 times $(p = \frac{1}{2} = q)$

$$P(3 \text{ heads}) = \binom{7}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^{7-3} = \binom{7}{3}\left(\tfrac{1}{2}\right)^7$$

$$= \binom{7}{3}\cdot\frac{1}{2^7} = \frac{7!}{3!\,4!}\cdot\frac{1}{128}$$

$$= \frac{7\cdot\cancel{6}\cdot 5}{\cancel{3!}}\cdot\frac{1}{128} = \frac{35}{128}$$

Negative Binomial distribution

Experiment: do Bernoulli trials (prob. p of success & q of failure) until we have k successes.

Q: What is the probability that this will require x trials?

P(x trials to get the k^{th} success)

$$= \binom{x-1}{k-1} p^k q^{x-k}$$

ways to pick which $k-1$ trials had the first $k-1$ successes

Example: Toss a fair coin until the 3^{rd} success

$$P(\text{you need 5 tosses}) = \binom{5-1}{3-1}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^{5-3}$$

$$= \binom{4}{2}\left(\tfrac{1}{2}\right)^5 = \frac{4!}{2!2!}\cdot\frac{1}{32}$$

$$= \frac{\cancel{4}\cdot 3\cdot \cancel{2}\cdot 1}{\cancel{2\cdot 1\cdot 2\cdot 1}}\cdot\frac{1}{\cancel{32}\,16} = \frac{3}{16}$$

Geometric distribution is the case $k=1$.

Note: $\binom{x-1}{1-1} = \binom{x-1}{0} = 1$