Discrete Probability continued:

Permutations & Combinations (§3.1 & 3.2)

Experiment: We have five coloured marbles:
B, G, Y, R, P

We draw the marbles at random from a box, without replacement, one at a time.

Outcomes: All the ways that the 5 marbles can be arranged (in a row).

Distribution: Uniform - each arrangement is as likely as any other.

So what is $m(\omega)$ for any arrangement ω?

$$\text{''} \quad \frac{1}{\#\text{ of arrangements}} = ? = \frac{1}{120}$$

# of arrangements = (# choices for first marbles)
$\cdot$ (— " —— second — " —)
$\cdot$ (— " —— third — " —)
$\cdot$ (— " —— fourth — " —)
$\cdot$ (— " —— fifth — " —)
$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$= 120$

In general, the number of ways to arrange n different things is

$$n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1$$

choices: 1^{st}, 2^{n}, 3^{rd}, …, $n-1^{st}$, n^{th}

$$= n! \qquad \text{"n factorial"} \qquad \left[\Gamma(n+1) = n!\right]$$

Gamma function

Alter the previous experiment:
Pick only two marbles.
(Still without replacement & one at a time.)

$$\#\,\text{outcomes} = (\#\text{ choices of first marble}) \cdot (\text{— '' — second — '' —})$$

$$= \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = \frac{5!}{3!} = \frac{5!}{(5-2)!}$$

In general, the number of ways to to arrange k objects from n different objects is

$$n(n-1)(n-2)\cdots\overbrace{(n-(k-1))}^{n-k+1} = \frac{n!}{(n-k)!}$$

choices: 1^{st}, 2^{nd}, 3^{rd}, …, k^{th}

<u>Notations</u>: $\frac{n!}{(n-k)!} = (n)_k$ (our text)

$$\left(= P^n_k = P_{n,k} = \cdots \right)$$

Experiment: Consider the 13 cards in ♡
(A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2)

Say we shuffle the cards thoroughly. We get a random arrangement of the thirteen cards.

Question: What is the probability that first five cards in the shuffled arrangement are 10, 9, 8, 7, 6 in that order?

outcomes of selecting 5 cards in order from 13

$$= (13)_5 = \frac{13!}{(13-5)!} = \frac{13!}{8!}$$

Probability of getting 10, 9, 8, 7, 6 is $\frac{1}{13!/8!} = \frac{8!}{13!} = \frac{1}{(13)_5}$.

Since $n!$ is very large and hard to compute, people use an approximation once n is too big for their calculator.

Stirling's Formula: When n is large

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n .$$

"Combinations": How many ways are there to choose k things from n things (if order doesn't matter)?

Compute "n choose k", $\binom{n}{k}$, by

$$\binom{n}{k} = \frac{\#\text{ ways to pick } k \text{ things from } n \text{ things in order}}{\#\text{ ways to arrange } k \text{ things}}$$

"binomial coefficient"

$$= \frac{n!/(n-k)!}{k!} = \frac{n!}{k!(n-k)!}$$

[Other notations: C^n_k, $C_{n,k}$, ...]

$$(a+b)^n = (a+b)(a+b)(a+b)\cdots(a+b) \quad \text{n copies of } a+b$$

$$= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

So to make the sum formula make sense we need $\binom{n}{0} = \binom{n}{n} = 1$. This needs

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1 = \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!}$$

so we <u>define $0! = 1$</u>.

Experiment: Draw a hand of five cards from a standard deck, at random.

Outcomes: The hands.

Distribution: $m(\text{hand}) = \dfrac{1}{\#\text{hands}} = \dfrac{1}{\binom{52}{5}} = \dfrac{5!\,47!}{52!}$

Q: $P(\text{4 of a kind in the hand}) = ?$ (4 As, or 4K's, ...)

$$P = \frac{\#\text{ hands with 4 of a kind}}{\#\text{ hands possible}}$$

$$= \frac{(\#\text{ choices for the kind})\cdot(\#\text{ choices for remaining card})}{\binom{52}{5}}$$

$$= \frac{\binom{13}{1}\cdot\binom{48}{1}}{\binom{52}{5}} = \frac{13\cdot 48}{\binom{52}{5}} = \ldots$$