Trent University, Summer 2017

MATH 1550H Test Solutions

Monday, 10 July

Time: 50 minutes

Name: ______________________________

Student Number: ______________________________

Question	**Mark**	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Determine whether $f(x) = \begin{cases} \frac{1}{2}e^{x} & x \leq 0 \\ \frac{1}{2}e^{-x} & x \geq 0 \end{cases}$ is a valid probability density.

b. A hand of five cards is drawn randomly, one at a time (so order matters) and without replacement, from a standard 52-card deck. What is the probability that the hand includes exactly three $\heartsuit$s, given that the first card drawn was a $\spadesuit$?

c. A fair coin is tossed until it comes up heads for the second time. What is the probability that at least four tosses will be required?

SOLUTIONS. **a.** We check that $f(x)$ satisfies the conditions needed to be a probability density function. First, since $e^t > 0$ for all $t \in \mathbb{R}$, it follows from the definition of $f(x)$ that $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Second, note that $\int e^x\,dx = e^x$, and that, using the substitution $u = -x$, so $du = (-1)\,dx$ and $dx = (-1)\,du$, $\int e^{-x}\,dx = \int e^u\,(-1)\,du = (-1)e^u = -e^{-x}$. It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_{0}^{\infty} \frac{1}{2}e^{-x}\,dx = \frac{1}{2}e^{x}\Big|_{-\infty}^{0} + \frac{1}{2}\left(-e^{-x}\right)\Big|_{0}^{\infty} \\ &= \left[\frac{1}{2}e^{0} - \frac{1}{2}e^{-\infty}\right] + \left[\frac{1}{2}\left(-e^{-\infty}\right) - \frac{1}{2}\left(-e^{-0}\right)\right] \\ &= \left[\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0\right] + \left[\frac{1}{2}\cdot 0 - \frac{1}{2}\cdot(-1)\right] = \frac{1}{2} + \frac{1}{2} = 1\,.\end{aligned}$$

Since it satisfies both conditions, $f(x)$ is indeed a valid probability density. $\square$

b. Let B be the event that the first card drawn was a $\spadesuit$, and let A be the event that there are exactly three $\heartsuit$s in the hand. We need to compute $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Note that there are $52\cdot 51\cdot 50\cdot 49\cdot 48 = \frac{52!}{47!}$ equally likely possible five-card hands if order matters and the cards are drawn without replacement. (52 choices for the first card in the hand, 51 choices left for the second card, and so on.)

To compute $P(B)$ we need to count the number of hands in B. There are 13 choices for the first card (since there are 13 $\spadesuit$s in the deck), 51 choices for the second card, 50 choices for the third card, 49 choices for the fourth card, and 48 choices for the fifth card. That is, there are $13\cdot 51\cdot 50\cdot 49\cdot 48$ hands in B, and so

$$P(B) = \frac{\#\text{ hands in } B}{\text{total } \#\text{ of hands}} = \frac{13\cdot 51\cdot 50\cdot 49\cdot 48}{52\cdot 51\cdot 50\cdot 49\cdot 48} = \frac{13}{52} = \frac{1}{4}\,.$$

To compute $P(A\cap B)$ we need to count the number of hands in $A\cap B$. There are 13 choices for the first card since it must be a $\spadesuit$ because $A\cap B \subseteq B$. On the other hand, because $A\cap B \subseteq A$, three of the remaining four cards must be $\heartsuit$s. There are $\binom{4}{3} = 4$ ways to pick which of the second through fifth cards are $\heartsuit$s, $13\cdot 12\cdot 11$ ways to put $\heartsuit$s in those three slots, and $52 - 13 - 1 = 38$ cards left in the deck for the remaining card which are neither $\heartsuit$s nor the $\spadesuit$ chosen for the first card. It follows that there are $13 \cdot 13 \cdot 12 \cdot 11 \cdot 38$ hands in $A \cap B$, so

$$P(A \cap B) = \frac{\#\text{ hands in } A \cap B}{\text{total } \#\text{ of hands}} = \frac{13 \cdot 13 \cdot 12 \cdot 11 \cdot 38}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}\,.$$

It follows that

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{\frac{13 \cdot 13 \cdot 12 \cdot 11 \cdot 38}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}}{\frac{13 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}} = \frac{13 \cdot 13 \cdot 12 \cdot 11 \cdot 38}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \cdot \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{13 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\ &= \frac{13 \cdot 13 \cdot 12 \cdot 11 \cdot 38}{13 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.0109\,. \quad \square \end{aligned}$$

c. The sample space here is $\Omega = \{HH, HTH, THH, HTTH, THTH, \dots\}$, with the probability function being given by $m\left(HT^kH\right) = \left(\frac{1}{2}\right)^{k+2}$. (Since the coin is fair, we have $P(H) = P(T) = \frac{1}{2}$ for each single toss of the coin.) It follows that

$$\begin{aligned} P(\geq 4 \text{ tosses}) &= 1 - P(< 4 \text{ tosses}) = 1 - P(HH,\ HTH, \text{ or } THH) \\ &= 1 - [m(HH) + m(HTH) + m(THH)] \\ &= 1 - \left[\frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right] 1 - \frac{1}{2} = \frac{1}{2}\,. \quad \blacksquare \end{aligned}$$

Note. **c** is also doable by observing that it involves a negative binomial distribution with $p = q = \frac{1}{2}$ and $k = 2$ successes, and getting the probability function that way instead of working it out from scratch.

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Suppose that A and B are events in some sample space, with $P(A) = P(B) = \frac{2}{3}$. What is the range of possible values of $P(A|B)$?

b. The continuous random variable X has the density function $g(x) = \begin{cases} \frac{2}{9}x & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$.
Compute the probability that $1 \leq X$, given that $X \leq 2$.

c. A fair standard six-sided die is rolled once. If it comes up with an odd number, it is rolled just one more time; if it comes up with an even number, it is not rolled again. Compute the probability that the last roll made came up with 1 or 5.

SOLUTIONS. **a.** Since $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and we are told that $P(B) = \frac{2}{3}$, the range of possible values of $P(A|B)$ depends on the range of possible values of $P(A \cap B)$. The most $P(A \cap B)$ could be is $\frac{2}{3}$, which would occur exactly when $A = A \cap B = B$. (Recall that $P(A) = \frac{2}{3}$, too.) The least that $P(A \cap B)$ could be is $\frac{1}{3}$, which would occur exactly when $P\left(A \cap \overline{B}\right) = P(A \cap B) = P\left(\overline{A} \cap B\right) = \frac{1}{3}$. (If we had $P(A \cap B) < \frac{1}{3}$, we would have $P\left(A \cap \overline{B}\right) = P\left(\overline{A} \cap B\right) = \frac{2}{3} - P(A \cap B)$, and so we would also have $P(A \cup B) = P\left(A \cap \overline{B}\right) + P(A \cap B) + P\left(\overline{A} \cap B\right) = \frac{4}{3} - P(A \cap B) > 1$, which is impossible.)

Since $\frac{1}{3} \leq P(A \cap B) \leq \frac{2}{3}$, we have $\frac{1}{2} = \frac{1/3}{2/3} \leq P(A|B) = \frac{P(A \cap B)}{P(B)} \leq \frac{2/3}{2/3} = 1$; that is, $P(A|B)$ must be between $\frac{1}{2}$ and 1, inclusive. □

b. The probability that $1 \leq X$, *given* that $X \leq 2$, is a conditional probability, namely $P(1 \leq X|X \leq 2) = \dfrac{P(1 \leq X \ \&\ X \leq 2)}{P(X \leq 2)} = \dfrac{P(1 \leq X \leq 2)}{P(X \leq 2)}$.

$$P(X \leq 2) = \int_{-\infty}^{2} g(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} \frac{2}{9}x\,dx = 0 + \frac{2}{9} \cdot \frac{x^2}{2}\bigg|_0^2 = \frac{x^2}{9}\bigg|_0^2$$
$$= \frac{2^2}{9} - \frac{0^2}{9} = \frac{4}{9} - 0 = \frac{4}{9}$$
$$P(1 \leq X \leq 2) = \int_{1}^{2} \frac{2}{9}x\,dx = \frac{2}{9} \cdot \frac{x^2}{2}\bigg|_1^2 = \frac{x^2}{9}\bigg|_1^2 = \frac{2^2}{9} - \frac{1^2}{9} = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

It follows that $P(1 \leq X|X \leq 2) = \dfrac{P(1 \leq X \leq 2)}{P(X \leq 2)} = \dfrac{1/3}{4/9} = \dfrac{1}{3} \cdot \dfrac{9}{4} = \dfrac{3}{4} = 0.75$. □

c. Since the die is fair, we have $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$ on any one roll. The only way to come up with a 1 or 5 on the *last* roll of the experiment is to do so on the second roll, because rolling a 1 or 5, which are odd numbers, on the first roll, requires you to roll again. It follows that

$$\begin{aligned} P(\text{1 or 5 on last roll}) &= P(\text{odd number on first roll})P(\text{1 or 5 on second roll}) \\ &= P(\text{roll 1, 3, or 5})P(\text{roll 1 or 5}) = \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right)\left(\frac{1}{6} + \frac{1}{6}\right) \\ &= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}. \quad \blacksquare \end{aligned}$$

3. Do *one (1)* of **a** or **b**. *[10]*

a. The continuous random variable W has the density function $h(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

Compute $P(W \geq 1)$.

b. A fair coin is tossed until it comes up heads for the third time. Let the random variable Y count the total number of tosses that occur in this experiment. Find the probability funtion of Y and compute $P(Y \leq 5)$.

Solutions. **a.** We first work out the antiderivative of xe^{-x}, which we will do using integration by parts. Let $u = x$ and $v' = e^{-x}$, so $u' = 1$ and $v = -e^{-x}$. (We have shown that $\int e^{-x}\,dx = -e^{-x}$ several times in class using the substitution $w = -x$.) Thus

$$\begin{aligned} \int xe^{-x}\,dx = \int uv'\,dx = uv - \int u'v\,dx = -xe^{-x} - \int 1 \cdot \left(-e^{-x}\right)\,dx \\ = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}\,. \end{aligned}$$

It folllows that

$$\begin{aligned} P(W \geq 1) &= \int_1^\infty h(x)\,dx = \int_1^\infty xe^{-x}\,dx = \left(-xe^{-x} - e^{-x}\right)\Big|_1^\infty \\ &= \left(-\infty e^{-\infty} - e^{-\infty}\right) - \left(-1e{-}1 - e^{-1}\right) = (0-0) - \left(-2e^{-1}\right) \\ &= 2e^{-1} = \frac{2}{e} \approx 0.7358\,. \end{aligned}$$

We were a little sloppy in throwing ∞ around here, but we get away with it here by remembering the fact that as you approach an infinity, exponential functions completely dominate polynomial functions. □

b. Y has a negative binomial distribution with $p = q = \frac{1}{2}$ (since the coin is fair) and $k = 3$ (since we go until we get the third head), so it has the probability distribution function $m(n) = \binom{n-1}{k-1} q^{n-k} p^k = \binom{n-1}{3-1} \left(\frac{1}{2}\right)^{n-3} \left(\frac{1}{2}\right) = \binom{n-1}{2} \left(\frac{1}{2}\right)^n$, where $n \geq k = 3$. It follows that

$$\begin{aligned} P(Y \leq 5) &= P(Y = 3) + P(Y = 4) + P(Y = 5) = m(3) + m(4) + m(5) \\ &= \binom{3-1}{2}\left(\frac{1}{2}\right)^3 + \binom{4-1}{2}\left(\frac{1}{2}\right)^4 + \binom{5-1}{2}\left(\frac{1}{2}\right)^5 \\ &= \binom{2}{2} \cdot \frac{1}{8} + \binom{3}{2} \cdot \frac{1}{16} + \binom{4}{2} \cdot \frac{1}{32} = 1 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + 6 \cdot \frac{1}{32} \\ &= \frac{2}{16} + \frac{3}{16} + \frac{3}{16} = \frac{8}{16} = \frac{1}{2} = 0.5\,. \quad \blacksquare \end{aligned}$$

[Total = 30]