Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2017

SOLUTIONS TO ASSIGNMENT #2
"The only way to win is cheat"

You run afoul of the Mathematical Inquisition unexpectedly*. Maybe you divided by zero, maybe you computed a derivative at a discontinuity, it doesn't matter. For whatever heinous offence you committed you are given an unfair but otherwise standard die, whose bias you do not know, and told to use it to solve the following tasks.

1. Use the die to simulate a fair coin. *[4]*

NOTE. That is, you need to figure out how to use the biased die – whose probabilities you do not know – in some process that has two equally likely. outcomes.

SOLUTION. Let E be the event that a sigle roll of the biased die comes up with an even-numbered face, and let O be the event that a single roll of the biased die comes up with an odd-numbered face. Since the die is biased and we don't know the bias, we cannot assume that $P(E) = P(O)$. However, if we toss the die twice, the two tosses are independent, so $P(EO) = P(E)P(O) = P(O)P(E) = P(OE)$. (Though we still don't know the actual value of $P(EO) = P(OE)$.)

The procedure for simulating a fair coin using the given biased die is as follows:

Roll the die twice.
If the outcome of the double roll was EO, the simulated coin came up H.
If the outcome of the double roll was OE, the simulated coin came up T.
If the outcome of the double roll was EE or OO, then repeat the process.

The probability of having to repeat the process $n > 1$ times is $[P(EE) + P(OO)]^n$. Since $P(EE) + P(OO) < 1$ [well, unless the die is so biased that $P(OE) + P(EO) = 0$], this probability goes to 0 as $n \to \infty$, so the probability that the process will run forever and never give a result is 0. Finally, since $P(EO) = P(OE)$, we have $P(H) = P(T)$, so the simulated coin is indeed fair. □

2. Use the die to simulate a fair standard die. *[3]*

SOLUTION. It's good enough to simulate the fair die with a fair coin, given that we know from answering the previous question how to simulated a fair coin using the given biased die.

The procedure for simulating a fair standard die using a fair coin is as follows:

Toss the fair coin three times.
If the outcome of the three tosses was HHH, then the simulated die came up 1.
If the outcome of the three tosses was HHT, then the simulated die came up 2.
If the outcome of the three tosses was HTH, then the simulated die came up 3.
If the outcome of the three tosses was THH, then the simulated die came up 4.
If the outcome of the three tosses was HTT, then the simulated die came up 5.
If the outcome of the three tosses was THT, then the simulated die came up 6.
If the outcome of the three tosses was TTH or TTT, then repeat the process.

* No one ever expects the Mathematical Inquisition!

The probability of having to repeat the process $n > 1$ times is $[P(TTH) + P(TTT)]^n = \left[\frac{1}{8} + \frac{1}{8}\right]^n = \left[\frac{1}{4}\right]^n$. This probability goes to 0 as $n \to \infty$, so the probability that the process will run forever and never give a result is 0. Finally, since the six outcomes assigned to the six faces of the simulated die are equally likely (because the coin is fair), it follows that the simulated die is fair. □

3. Use the die to simulate a biased coin with $P(H) = \dfrac{41}{123}$ and $P(T) = \dfrac{82}{123}$. *[3]*

Solution. We'll make our lives a bit easier by observing that $P(H) = \dfrac{41}{123} = \dfrac{1}{3}$ and $P(T) = \dfrac{82}{123} = \dfrac{2}{3}$. Note that it is good enough to simulate the biased coin using a fair die, given that we know from the answer to the previous problem how to simulate a fair die using the given biased die.

The procedure for simulating the biased coin using a fair die is as follows:

Roll the fair die once.
If the face that came up is 1 or 2, then the simulated coin came up H.
If the face that came up is 3, 4, 5, or 6, the the simulated coin came up T.

Since the die is fair, $P(H) = P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ and $P(T) = P(3, 4, 5, \text{ or } 6) = P(3) + P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$, as required.

Note that given the answers to the previous questions, there is no need to worry about this procedure running on forever, as that problem was met and dealt with when using the biased die to simulate a fair die. ■