Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2017

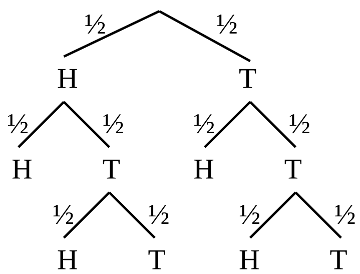
Solutions to the Quizzes

Quiz #1. Wednesday, 21 June, 2017. *[10 minutes]*

Consider the following experiment. A fair coin is tossed twice. If it comes up heads on the second toss, the experiment ends; if it comes up tails on the second toss, the coin is tossed just one more time, ending the experiment.

1. List all the outcomes in the sample space Ω for this experiment. *[1.5]*
2. What is the probability distribution function for this experiment? *[1.5]*
3. Let E be the event that an odd number of heads came up in the experiment. Compute the probability of E. *[2]*

SOLUTIONS. Just for the heck of it, here is the tree diagram for this experiment:



It may make it a bit easier to see how the experiment works.

1. $\Omega = \{ HH, HTH, HTT, TH, TTH, TTT \}$. $\square$

2. Using the tree diagram to help work out the probabilities, we have $p(HH) = p(TH) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $p(HTH) = p(HTT) = p(TTH) = p(TTT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. $\square$

3. As a collection of outcomes, the event in question is $E = \{ HTT, TH, TTH \}$, so its probability is $P(E) = p(HTT) + p(TH) + p(TTH) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$. $\blacksquare$

Quiz#2. Monday, 26 June, 2017. *[12 minutes]*

Consider the following experiment. A fair coin is tossed once, and then tossed repeatedly until the face that came up on the first toss comes up again.

1. What is the sample space for this experiment? *[1.5]*
2. What is the probability distribution function for this experiment? *[1.5]*
3. Let X count the total number of tosses made in the experiment. What is the probability that $X \geq 5$? *[2]*

SOLUTIONS. 1. $\Omega = \{\, HH, TT, HTH, THT, HTTH, THHT, HTTTH, THHHT, \dots \}$ $\square$

2. $p(HH) = P(TT) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $p(HTH) = p(THT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, and so on. In general, $p\left(HT^kH\right) = p\left(TH^kT\right) = \left(\frac{1}{2}\right)^{k+2} = \frac{1}{2^{k+2}}$. $\square$

3. *(The finite way.)* Here goes:

$$\begin{aligned} P(X \geq 5) &= 1 - P(X < 5) = 1 - [P(X = 2) + P(X = 3) + P(X - 4)] \\ &= 1 - [p(HH) + p(TT) + p(HTH) + p(THT) + p(HTTH) + p(THHT)] \\ &= 1 - \left[\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}\right] = 1 - \frac{7}{8} = \frac{1}{8} \qquad \square \end{aligned}$$

3. *(The way of series.)* Here goes, with the help of the summation formula for geometric series:

$$\begin{aligned} P(X \geq 5) &= P(X = 5) + P(X = 6) + P(X = 7) + \cdots \\ &= [p(HTTTH) + p(THHHT)] + [p(HTTTTH) + p(THHHH)] \\ &\quad + [p(HTTTTTH) + p(THHHHHT)] + \cdots \\ &= \left[\frac{1}{32} + \frac{1}{32}\right] + \left[\frac{1}{64} + \frac{1}{64}\right] + \left[\frac{1}{128} + \frac{1}{128}\right] + \cdots \\ &= \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = \frac{\frac{1}{16}}{1 - \frac{1}{2}} = \frac{\frac{1}{16}}{\frac{1}{2}} = \frac{1}{16} \cdot \frac{2}{1} = \frac{1}{8} \qquad \blacksquare \end{aligned}$$

Quiz #3. Wednesday, 28 June, 2017. *[10 minutes]*

1. A hand of five cards is drawn simultaneously [so without order or replacement] from a well-shuffled standard deck. What is the probability that the hand has "two pairs" [*i.e.* two cards of one kind, two cards of a different kind, and one card of yet another kind]?

SOLUTION. There are $\binom{13}{2} = 78$ ways to choose the kinds of the two pairs, $\binom{4}{2} = 6$ ways to choose the pair for each kind, $\binom{11}{1} = 11$ ways to choose the kind of the remaining card, and $\binom{4}{1} = 4$ ways to choose the card of that kind. It follows that there are $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1} = 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4 = 123552$ "two pair" hands. Since there are $\binom{52}{5} = 2598960$ equally likely hands, the probability that a randomly drawn hand has "two pairs" is $\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{123552}{2598960} \approx 0.0475$. $\blacksquare$

QUESTION. Why isn't the number of "two pair" hands $\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$?

ANSWER. Because the order that the kinds of the pairs come in doesn't matter.

Quiz #4. Wednesday, 5 July, 2017. *[15 minutes]*

The continuous random variable X has $f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Let A be the event that $X \leq \frac{1}{2}$ and let B be the event that $\frac{1}{4} \leq X$.

1. Compute $P(A|B)$, the conditional probability of A given B. *[5]*

Solution. We shall need calculus. (Unless we happen to be Archimedes ... :-) In particular, we will need the antiderivative of $3x^2$. Using the Power Rule for integration,

$$\int 3x^2\,dx = 3\int x^2\,dx = 3 \cdot \frac{x^{2+1}}{2+1} = 3 \cdot \frac{x^3}{3} = x^3\,.$$

[Technically, there should be a constant of integration, usually denoted by C, added to the antiderivative. Since we will only use the antiderivative to compute definite integrals, we can ignore the constant as it will always cancel out. If we were solving differential equations, on the other hand, we would use the initial conditions for the equations to determine the value of C.]

We can now compute $P(B)$ and $P(A \cap B)$:

$$\begin{aligned}
P(B) &= P\left(\frac{1}{4} \leq X\right) = \int_{1/4}^{\infty} f(x)\,dx = \int_{1/4}^{1} 3x^2\,dx + \int_{1}^{\infty} 0\,dx \\
&= x^3\Big|_{1/4}^{1} + 0 = 1^3 - \left(\frac{1}{4}\right)^3 = 1 - \frac{1}{64} = \frac{63}{64} = 0.984375 \\
P(A \cap B) &= P\left(X \leq \frac{1}{2} \text{ and } \frac{1}{4} \leq X\right) = P\left(\frac{1}{4} \leq X \leq \frac{1}{2}\right) \\
&= \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 3x^2\,dx = x^3\Big|_{1/4}^{1/2} \\
&= \left(\frac{1}{2}\right)^3 - \left(\frac{1}{4}\right) = \frac{1}{8} - \frac{1}{64} = \frac{8}{64} - \frac{1}{64} = \frac{7}{64} = 0.109375
\end{aligned}$$

By the definition of $P(A|B)$, it follows that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{7}{64}}{\frac{63}{64}} = \frac{7}{64} \cdot \frac{64}{63} = \frac{7}{63} = \frac{1}{9} \approx 0.111111\,. \quad \blacksquare$$

Quiz #5. Wednesday, 12 July, 2017. *[10 minutes]*

Do *one* (1) of the following questions.

1. The continuous random variable X has $f(x) = \begin{cases} 1 - \frac{1}{2}x & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Find the expected value, $E(X)$, of X. *[5]*

2. A fair coin is tossed until it comes up heads or until three tosses have been made, whichever comes first. The random variable Y counts how many tosses came up tails in the experiment. Find the expected value, $E(Y)$, of Y. *[5]*

SOLUTIONS. 1. *(Corrected.)* By definition,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{2} x \left(1 - \frac{1}{2}x\right) dx + \int_{2}^{\infty} x \cdot 0\, dx \\ &= 0 + \int_0^2 \left(x - \frac{1}{2}x^2\right) dx + 0 = \left(\frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^3}{3}\right)\Big|_0^2 \\ &= \left(\frac{2^2}{2} - \frac{1}{2} \cdot \frac{2^3}{3}\right) - \left(\frac{0^2}{2} - \frac{1}{2} \cdot \frac{0^3}{3}\right) = \left(2 - \frac{4}{3}\right) - 0 = \frac{2}{3} \approx 0.6667\,. \qquad \square \end{aligned}$$

2. The underlying sample space is $\Omega = \{H, TH, TTH, TTT\}$, and, since the coin is fair, $P(H) = \frac{1}{2}$, $P(TH) = \frac{1}{4}$, and $P(TTH) = P(TTT) = \frac{1}{8}$. If Y counts the number of tosses that came up tails in the experiment, then the possible values of Y are 0, 1, 2, and 3, with $P(Y = 0) = P(H) = \frac{1}{2}$, $P(Y = 1) = P(TH) = \frac{1}{4}$, $P(Y = 2) = P(TTH) = \frac{1}{8}$, and $P(Y = 3) = P(TTT) = \frac{1}{8}$. By definition, it follows that

$$\begin{aligned} E(Y) &= \sum_{y=0}^{3} y P(Y = y) = 0P(Y = 0) + 1P(Y = 1) + 2P(Y = 2) + 3P(Y = 3) \\ &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} = 0 + \frac{1}{4} + \frac{1}{4} + \frac{3}{8} = \frac{7}{8} = 0.875\,. \qquad \blacksquare \end{aligned}$$

Quiz #6. Monday, 17 July, 2017. *[15 minutes]*

Do *one* (1) of the following questions.

1. The continuous random variable X has $g(x) = \begin{cases} \frac{3}{4}\left(1-x^2\right) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Find the variance, $V(X)$, of X. *[5]*
2. A fair four-sided die with faces numbered 1 through 4 is rolled twice. The discrete random variable W records the sum of the faces. Find the the variance, $V(W)$, of W. *[5]*

SOLUTIONS. 1. To find $V(X) = E\left(X^2\right) - [E(X)]^2$ we need to compute both $E(X)$ and $E\left(X^2\right)$:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xg(x)\,dx = \int_{-\infty}^{-1} x \cdot o\,dx + \int_{-1}^{1} x \cdot \frac{3}{4}\left(1-x^2\right)\,dx + \int_{1}^{\infty} x \cdot 0\,dx \\ &= 0 + \frac{3}{4}\int_{-1}^{1}\left(x - x^3\right)\,dx + 0 = \frac{3}{4}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{-1}^{1} \\ &= \frac{3}{4}\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \frac{3}{4}\left(\frac{(-1)^2}{2} - \frac{(-1)^4}{4}\right) = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{3}{4}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{16} - \frac{3}{16} = 0 \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 g(x)\,dx = \int_{-\infty}^{-1} x^2 \cdot o\,dx + \int_{-1}^{1} x^2 \cdot \frac{3}{4}\left(1-x^2\right)\,dx + \int_{1}^{\infty} x^2 \cdot 0\,dx \\ &= 0 + \frac{3}{4}\int_{-1}^{1}\left(x^2 - x^4\right)\,dx + 0 = \frac{3}{4}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-1}^{1} \\ &= \frac{3}{4}\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \frac{3}{4}\left(\frac{(-1)^3}{3} - \frac{(-1)^5}{5}\right) = \frac{3}{4}\left(\frac{1}{3} - \frac{1}{5}\right) - \frac{3}{4}\left(\frac{-1}{3} - \frac{-1}{5}\right) \\ &= \frac{3}{4} \cdot \frac{2}{15} - \frac{3}{4} \cdot \frac{-2}{15} = \frac{1}{10} - \frac{-1}{10} = \frac{2}{10} = \frac{1}{5} \end{aligned}$$

It follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{1}{5} - 0^2 = \frac{1}{5} = 0.2$. $\square$

2. The underlying sample space is $\Omega = \{\,(1,1), (1,2), \dots, (4,3), (4,4)\,\}$, and since the die is fair, any outcome is as likely as any other, so $m(\omega) = \frac{1}{4 \cdot 4} = \frac{1}{16}$ for every outcome $\omega \in \Omega$. Thus the possible values of W are $2 = 1+1$ through $8 = 4+4$, and their probabilities are given by:

w	$P(W = w)$
2	$m(1,1) = \frac{1}{16}$
3	$m(1,2) + m(2,1) = \frac{2}{16}$
4	$m(1,3) + m(2,2) + m(3,1) = \frac{3}{16}$
5	$m(1,4) + m(2,3) + m(3,2) + m(4,1) = \frac{4}{16}$
6	$m(2,4) + m(3,3) + m(4,2) = \frac{3}{16}$
7	$m(3,4) + m(4,3) = \frac{2}{16}$
8	$m(4,4) = \frac{1}{16}$

To find $V(W) = E\left(W^2\right) - [E(W)]^2$ we need to compute both $E(W)$,

$$\begin{aligned} E(W) &= \sum_{w=2}^{8} wP(W = w) = 2 \cdot \frac{1}{16} + 3 \cdot \frac{2}{16} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{4}{16} + 6 \cdot \frac{3}{16} + 7 \cdot \frac{2}{16} + 8 \cdot \frac{1}{16} \\ &= \frac{2+6+12+20+18+14+8}{16} = \frac{80}{16} = 5\,, \end{aligned}$$

and $E\left(W^2\right)$,

$$
\begin{aligned}
E\left(W^2\right) &= \sum_{w=2}^{8} w^2 P(W=w) = 2^2 \cdot \frac{1}{16} + 3^2 \cdot \frac{2}{16} + 4^2 \cdot \frac{3}{16} + 5^2 \cdot \frac{4}{16} + 6^2 \cdot \frac{3}{16} + 7^2 \cdot \frac{2}{16} + 8^2 \cdot \frac{1}{16} \\
&= \frac{4+18+48+100+108+98+64}{16} = \frac{440}{16} = \frac{55}{2} = 27.5\,.
\end{aligned}
$$

Then $V(X) = E\left(X^2\right) - [E(X)]^2 = 27.5 - 5^2 = 27.5 - 25 = 2.5 = \frac{5}{2}$. ■

Quiz #7. Wednesday, 19 July, 2017. *[15 minutes]*

The continuous random variables X_1, X_2, X_3, and X_4 are independently and identically distributed with a normal distribution that has expected value $\mu = 3$ and standard deviation $\sigma = 4$. Let $\bar{X} = \dfrac{X_1 + X_2 + X_3 + X_4}{4}$; then $\bar{X}$ also has a normal distribution with expected value $\mu = 3$ (albeit it has a different variance and standard deviation).

1. Compute $P\left(-2 \le X_1 \le 7\right)$. *[2]*
2. Compute $P\left(-2 \le \bar{X} \le 7\right)$. *[3]*

Solutions. 1. X_1 has a normal distribution with expected value $\mu = 3$ and standard deviation $\sigma = 4$, so $Z = \frac{X_1 - 3}{4}$ has a standard normal distribution. Since

$$-2 \le X_1 \le 7 \Leftrightarrow -5 = -2 - 3 \le X_1 - 3 \le 7 - 3 = 4 \Leftrightarrow -1.25 = -\frac{5}{4} \le Z = \frac{X_1 - 3}{4} \le \frac{4}{4} = 1\,,$$

it follows, with the help of the cumulative standard normal table, that

$$\begin{aligned} P\left(-2 \le X_1 \le 7\right) &= P(-1.25 \le Z \le 1) = P(Z \le 1) - P(Z \le -1.25) \\ &= 0.8413 - 0.1056 = 0.7357\,. \qquad \square \end{aligned}$$

2. We need to work out the standard deviation of $\bar{X}$ before we can apply the procedure used in answering the previous question. The variance of each X_i is $V\left(X_i\right) = \sigma^2 = 4^2 = 16$. Since the X_i are independently and identically distributed, we have:

$$\begin{aligned} V\left(\bar{X}\right) &= V\left(\frac{1}{4}\left[X_1 + X_2 + X_3 + X_4\right]\right) = \left(\frac{1}{4}\right)^2 V\left(X_1 + X_2 + X_3 + X_4\right) \\ &= \frac{1}{16}\left[V\left(X_1\right) + V\left(X_2\right) + V\left(X_3\right) + V\left(X_4\right)\right] = \frac{1}{16} \cdot 4 \cdot 16 = 4 \end{aligned}$$

It follows that the standard deviation of $\bar{X}$ is $\sigma_{\bar{X}} = \sqrt{V\left(\bar{X}\right)} = \sqrt{4} = 2$.

Since $\bar{X}$ has a normal distribution with expected value $\mu_{\bar{X}} = 3$ and standard deviation $\sigma_{\bar{X}} = 2$, $\bar{Z} = \frac{\bar{X} - 3}{2}$ has a standard normal distribution. Since

$$-2 \le \bar{X} \le 7 \Leftrightarrow -5 = -2 - 3 \le \bar{X} - 3 \le 7 - 3 = 4 \Leftrightarrow -2.5 = -\frac{5}{2} \le \bar{Z} = \frac{\bar{X} - 3}{2} \le \frac{4}{2} = 2\,,$$

it follows, with the help of the cumulative standard normal table, that

$$\begin{aligned} P\left(-2 \le \bar{X} \le 7\right) &= P\left(-2.5 \le \bar{Z} \le 2\right) = P\left(\bar{Z} \le 2\right) - P\left(\bar{Z} \le -2.5\right) \\ &= 0.9772 - 0.0062 = 0.9710\,. \qquad \blacksquare \end{aligned}$$