Mathematics 1550H – Introduction to probability
Trent University, Summer 2017

Final Examination
Saturday, 29 July, 2017

Spatio-temporal locus: 14:00–17:00 in FPHL 117 *Inflicted by* Стефан Біланюк.

Instructions: Do both of parts **Card** and **Coin**, and, if you wish, part **Die**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; $8.5'' \times 11''$ or A4 aid sheet; standard normal table; one brain (caffeine optional).

Part Card. Do all of **1–5**. *[Subtotal = 68/100]*

1. A hand of five cards is drawn randomly, simultaneously and without replacement, from a standard 52-card deck.

a. What is the probability that all the cards in the hand are $\diamondsuit$s and/or $\clubsuit$s? *[5]*

b. What is the probability that the hand is a *full house*, consisting of three of one kind and two of another kind? *[5]*

c. What is the probability that the hand would be counted in both **a** and **b**? *[5]*

Solutions. **a.** There are $\binom{52}{5}$ different (unordered) five-card hands that can be drawn from a standard 52-card deck, each equally likely. Since there are thirteen cards in each suit, there are $26 = 13 + 13$ cards in the deck that are $\diamondsuit$s or $\clubsuit$s, and hence $\binom{26}{5}$ possible hands that are drawn entirely from these two suits. It follows that the probability that all the cards in a randomly drawn hand are $\diamondsuit$s and/or $\clubsuit$s is $\binom{26}{5}/\binom{52}{5}$. $\square$

b. There are thirteen kinds and four cards, one of each suit, of each kind. It follows that there are $\binom{13}{1}$ ways to choose a kind for the three-of-a-kind and $\binom{4}{3}$ ways to choose the three of that kind, and then there are $\binom{12}{1}$ ways left to choose the kind for the two-of-a-kind and $\binom{4}{2}$ ways to choose the two of that kind. Thus there are $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ hands that are a full house. Since there are $\binom{52}{5}$ possible equally likely hands, the probability of drawing a full house is $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}/\binom{52}{5}$. $\square$

c. This question boils down to counting how many full house hands are drawn entirely from $\diamondsuit$s and/or $\clubsuit$s. There are, in fact, no such hands. A full house includes three-of-a-kind, and a three-of-a-kind must have cards from three different suits, so they cannot all be $\diamondsuit$s and/or $\clubsuit$s. Hence the probability that a five-card hand satisfies the conditions given in both **a** and **b** is zero. $\blacksquare$

2. Let W be a continuous random variable with the following probability density function:

$$g(w) = \begin{cases} w^{-2} & w \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $g(w)$ is indeed a probability density function. *[8]*

b. Compute the probability that $W \geq 4$, given that $W \geq 2$. *[7]*

c. Find the expected value, $E(W)$, of W. *[5]*

SOLUTIONS. **a.** Since $w^{-2} = \frac{1}{w^2} > 0$ for all $w \geq 1$ and $0 = 0$ for all $w < 1$, we have that $g(w) \geq 0$ for all w. As

$$\begin{aligned} \int_{-\infty}^{\infty} g(w)\, dw &= \int_{-\infty}^{1} 0\, dw + \int_{1}^{\infty} w^{-2}\, dw = 0 + \left.\frac{w^{-2+1}}{-2+1}\right|_1^\infty = \left. -w^{-1}\right|_1^\infty \\ &= \left. -\frac{1}{w}\right|_1^\infty = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{1}\right) = -0 - (-1) = 1\,, \end{aligned}$$

it follows that $g(w)$ is a valid probability density function.

b. The probability that $W \geq 4$, given that $W \geq 2$, is the conditional probability

$$P(W \geq 4 | W \geq 2) = \frac{P(W \geq 4 \ \&\ W \geq 2)}{P(W \geq 2)} = \frac{P(W \geq 4)}{P(W \geq 2)}\,,$$

so we compute $P(W \geq 4)$ and $P(W \geq 2)$; note that we already worked out that $\int w^{-2}\, dw = -\frac{1}{w}$ in the solution to **a** above.

$$\begin{aligned} P(W \geq 4) &= \int_4^\infty g(w)\, dw = \int_4^\infty w^{-2}\, dw = \left. -\frac{1}{w}\right|_4^\infty = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{4}\right) = 0 + \frac{1}{4} = \frac{1}{4} \\ P(W \geq 2) &= \int_2^\infty g(w)\, dw = \int_2^\infty w^{-2}\, dw = \left. -\frac{1}{w}\right|_2^\infty = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2} \end{aligned}$$

It follows that $P(W \geq 4 | W \geq 2) = \frac{P(W \geq 4 \ \&\ W \geq 2)}{P(W \geq 2)} = \frac{P(W \geq 4)}{P(W \geq 2)} = \frac{1/4}{1/2} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$. □

c. By definition,

$$\begin{aligned} E(W) &= \int_{-\infty}^{\infty} w \cdot g(w)\, dw = \int_{-\infty}^{1} w \cdot 0\, dw + \int_1^\infty w \cdot w^{-2}\, dw = 0 + \int_1^\infty w^{-1}\, dw \\ &= \left. \ln(w)\right|_1^\infty = \ln(\infty) - \ln(1) = \infty - 0 = \infty\,. \end{aligned}$$

That is, $g(w)$ is one of those unfortunate probability densities for which expected value is not well-defined. ■

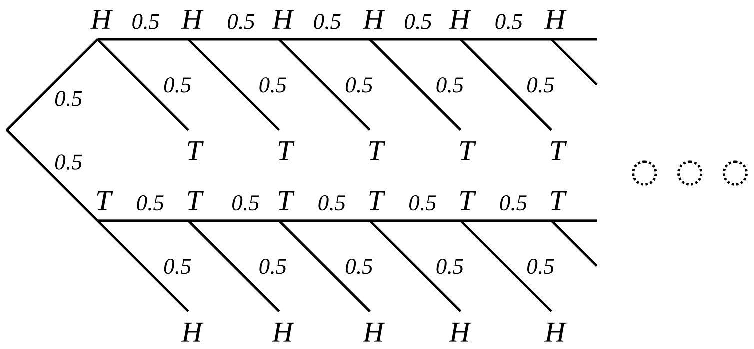
3. A fair coin is tossed, and then tossed repeatedly until it comes up with a face different from the one that came up on the first toss.

a. Draw the tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Let the random variable X count the total number of tosses that occur in the experiment. Find the expected value $E(X)$ and variance $V(X)$ of X. *[7]*

SOLUTIONS. **a.** Here is an initial piece of the infinite tree:



$\square$

b. The sample space for the experiment is $\Omega = \{HT, TH, HHT, TTH, HHHT, \dots\}$ and the probability function is given by $m\left(H^kT\right) = m\left(T^kH\right) = \left(\frac{1}{2}\right)^k \frac{1}{2} = \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2^{k+1}}$, where $k \geq 1$. $\square$

c. Note that $X-1$ has a geometric distribution with probability of success $p = \frac{1}{2}$ (because the coin is fair): the first toss defines success as being the face other than the one came up, and one then tosses the coin until success is achieved. It follows that $E(X-1) = \frac{1}{p} = \frac{1}{1/2} = 2$, so $E(X) = E(X-1+1) = E(X-1) + E(1) = 2+1 = 3$. Similarly, $V(X-1) = \frac{q}{p^2} = \frac{1/2}{(1/2)^2} = \frac{1}{1/2} = 2$. Since the probability of X having some value is unaffected by the value of 1, X and the constant "random" variable 1 are independent, so $V(X) = V(X-1+1) = V(X-1) + V(1) = 2+0 = 2$. (A constant "random" variable does not vary and has variance 0 ...) $\blacksquare$

4. The continuous random variable Y has an exponential distribution with variance $V(Y) = 4$. What is the probability density function of Y? *[5]*

SOLUTION. An exponential distribution with parameter $\lambda > 0$ has density function $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ and variance $\frac{1}{\lambda^2}$. Since $\frac{1}{\lambda^2} = V(Y) = 4$ in this case, we have $4\lambda^2 = 1$ and so $\lambda^2 = \frac{1}{4}$, from which it follows that $\lambda = \sqrt{\frac{1}{4}} = \frac{1}{2}$. Thus the probability density function of Y is $f(t) = \begin{cases} \frac{1}{2}e^{-t/2} & t \geq 0 \\ 0 & t < 0 \end{cases}$. $\blacksquare$

5. Suppose X is a continuous random variable that has a normal distribution with expected value $\mu = -2$ and standard deviation $\sigma = 5$.

a. Compute $P(1 \leq X \leq 5)$ with the help of a standard normal table. *[6]*

b. Use Chebyshev's Inequality to get as small an upper bound for $P(X \geq 8)$ as you can. *[7]*

SOLUTIONS. **a.** Since $Z = \frac{X-\mu}{\sigma} = \frac{X-(-2)}{5}$ has a standard normal distribution and

$$\begin{aligned} 1 \leq X \leq 5 &\Leftrightarrow 3 = 1-(-2) \leq X-(-2) \leq 5-(-2) = 7 \\ &\Leftrightarrow 0.6 = \frac{3}{5} \leq Z = \frac{X-(-2)}{5} \leq \frac{7}{5} = 1.4\,, \end{aligned}$$

we have $P(1 \leq X \leq 5) = P(0.6 \leq Z \leq 1.4) = P(Z \leq 1.4) - P(Z \leq 0.6) \approx 0.9192 - 0.7257 = 0.1935$. $\square$

b. Since

$$X \geq 8 \Leftrightarrow X-(-2) \geq 8-(-2) = 10 \Rightarrow |X-(-2)| \geq 10\,,$$

Chebyshev's Inequality tells us that

$$P(X \geq 8) \leq P\left(|X-(-2)| \geq 10\right) \leq \frac{5^2}{10^2} = \frac{25}{100} = \frac{1}{4} = 0.25\,. \quad \blacksquare$$

[Parts **Coin** *and* **Die** *are on page 2.]*

Part Coin. Do any *two* (2) of **6–9**. *[Subtotal = 32/100]*

6. You are given two bowls and 180 marbles, 100 of them black, 50 of them white, and 30 of them green. A blindfolded assistant will select a bowl at random, and then select a marble at random from that bowl. How should you distribute the marbles between the two bowls to maximize the probability of the assistant selecting a white marble? Provide as complete an explanation as you can. *[16]*

SOLUTION. You should put a single white marble in one bowl and all 179 of the other marbles in the other bowl. If the assistant chooses the former bowl, they are guaranteed to select the white marble in it; if they choose the other bowl, there is still a 49 out 179 chance they will then select a white a marble. Since the bowls are equally likely to be selected, there is a total probability of $\frac{1}{2} + \frac{1}{2} \cdot \frac{49}{179} = \frac{114}{179} \approx 0.6369$ that the assistant will select a white marble.

Why is this the best that can be done? Experiment a bit with moving marbles from the bin with nearly all of them to the one with the single white marble. It is not hard to convince yourself that every way of doing this will reduce the overall probability of getting a white marble, except for moving all the marbles but one white one over, which gets you back into the same situation with the roles of the bowls reversed. ■

7. A fair coin is tossed ten times. The random variable X counts how many pairs of consecutive tosses had the same face come up.

a. What are the possible values and probability function for X? *[8]*

b. Find the expected value $E(X)$ and variance $V(X)$ of X. *[8]*

SOLUTIONS. **a.** There are nine pairs of consecutive tosses: tosses 1 and 2, 2 and 3, and so on, through tosses 9 and 10. Whether the first toss comes up heads or tails, the second toss has a probability of $P(H) = P(T) = \frac{1}{2}$ of coming up with the same face. (Alternatively, each pair has a probability of $P(HH) + P(TT) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ of coming up with the same face.) X therefore counts the number of successes in nine Bernoulli trials woth probability of success $p = \frac{1}{2}$; *i.e.* X has a binomial distribution with $n = 9$, $p = \frac{1}{2}$, and $q = 1 - p = \frac{1}{2}$. It follows that the possible values of X are 0, 1, ... , and 9, and the probability function of X is $p(k) = \binom{n}{k} p^k q^{n-k} = \binom{9}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{9-k} = \binom{9}{k} = \left(\frac{1}{2}\right)^9$. □

b. X has a binomial distribution with $n = 9$, $p = \frac{1}{2}$, and $q = 1 - p = \frac{1}{2}$, so X has expected value $E(X) = np = 9 \cdot \frac{1}{2} = \frac{9}{2} = 4.5$ and variance $V(X) = npq = 9 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{9}{4} = 2.25$. ■

8. Suppose that $g(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \le x < 0 \\ \frac{1}{2}e^{-x} & x \ge 0 \end{cases}$ is the probability density function of the continuous random variable X.

a. Verify that $g(x)$ is indeed a probability density function. *[6]*

b. Compute the expected value $E(X)$ and variance of $V(X)$ of X. *[10]*

SOLUTIONS. **a.** When $x < -1$, $0 \ge 0$, when $-1 \le x < 0$, $x + 1 \ge (-1) + 1 = 0$, and when $x \ge 0$, $\frac{1}{2}e^{-x} > 0$, so it follows that $g(x) \ge x$ for all x. Since we also have

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} (x+1)\,dx + \int_{1}^{\infty} \frac{1}{2}e^{-x}\,dx & &\text{Let } u = -x, \text{ so } du = (-1)\,dx \text{ and } dx = (-1)\,du, \\ &= 0 + \left(\frac{x^2}{2} + x\right)\Big|_{-1}^{0} + \frac{1}{2}\int_{0}^{-\infty} e^u (-1)\,du & &\text{and } \begin{smallmatrix} x & 0 & \infty \\ u & 0 & -\infty \end{smallmatrix}. \\ &= \left(\frac{0^2}{2} + 0\right) - \left(\frac{(-1)^2}{2} + (-1)\right) + \frac{1}{2}(-1)e^u\Big|_{0}^{-\infty} \\ &= 0 - \left(-\frac{1}{2}\right) + \frac{1}{2}(-1)e^{-\infty} - \frac{1}{2}(-1)e^0 = \frac{1}{2} - \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2} + 0 + \frac{1}{2} = 1\,, \end{aligned}$$

$g(x)$ is indeed a valid probability density function. $\square$

b. As worked out in the solution to **a** above, $\int e^{-x}\,dx = -e^{-x}$. Using this and integration by parts with $u = x$ and $v' = e^{-x}$, so $u' = 1$ and $v = -e^{-x}$, we get

$$\begin{aligned} \int xe^{-x}\,dx &= \int uv'\,dx = uv - \int u'v\,dx = x\left(-e^{-x}\right) - \int 1\left(-e^{-x}\right)\,dx \\ &= -xe^{-x} + \left(-e^{-x}\right) = -xe^{-x} - e^{-x}\,. \end{aligned}$$

Similarly, using the above and integration by parts with $u = x^2$ and $v' = e^{-x}$, so $u' = 2x$ and $v = -e^{-x}$, we get

$$\begin{aligned} \int x^2e^{-x}\,dx &= \int uv'\,dx = uv - \int u'v\,dx = x^2\left(-e^{-x}\right) - \int 2x\left(-e^{-x}\right)\,dx \\ &= -x^2e^{-x} + 2\left[-xe^{-x} - e^{-x}\right] = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}\,. \end{aligned}$$

Calculus preliminaries done! On to the main events:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xg(x)\,dx = \int_{-\infty}^{-1} x\cdot 0\,dx + \int_{-1}^{0} x(x+1)\,dx + \int_{0}^{\infty} x\cdot\frac{1}{2}e^{-x}\,dx \\ &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} \left(x^2 + x\right)\,dx + \frac{1}{2}\int_{0}^{\infty} xe^{-x}\,dx \\ &= 0 + \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{0} + \frac{1}{2}\left(-xe^{-x} - e^{-x}\right)\Big|_{0}^{\infty} \\ &= \left(\frac{0^3}{3} + \frac{0^2}{2}\right) - \left(\frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right) + \frac{1}{2}\left(-\infty e^{-\infty} - e^{-\infty}\right) - \frac{1}{2}\left(-0e^{-0} - e^{-0}\right) \\ &= 0 - \left(-\frac{1}{3} + \frac{1}{2}\right) + \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot(0-1) = -\left(-\frac{1}{6}\right) + 0 - \frac{1}{2}(-1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3} \end{aligned}$$

To compute $V(X)$ we will need to compute $E\left(X^2\right)$.

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 g(x)\,dx = \int_{-\infty}^{-1} x^2 \cdot 0\,dx + \int_{-1}^{0} x^2(x+1)\,dx + \int_{0}^{\infty} x^2 \cdot \frac{1}{2}e^{-x}\,dx \\
&= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} \left(x^3+x^2\right)\,dx + \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}\,dx \\
&= 0 + \left(\frac{x^4}{4}+\frac{x^3}{3}\right)\Bigg|_{-1}^{0} + \frac{1}{2}\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)\Bigg|_{0}^{\infty} \\
&= \left(\frac{0^4}{4}+\frac{0^3}{3}\right) - \left(\frac{(-1)^4}{4}+\frac{(-1)^3}{3}\right) + \frac{1}{2}\left(-x^2e^{-\infty} - 2\infty e^{-\infty} - 2e^{-\infty}\right) \\
&\quad - \frac{1}{2}\left(-0^2e^{-0} - 2\cdot 0e^{-0} - 2e^{-0}\right) \\
&= 0 - \left(\frac{1}{4} - \frac{-1}{3}\right) + \frac{1}{2}\cdot 0 - \frac{1}{2}(-0-0-2\cdot 1) = -\left(\frac{1}{4}-\frac{1}{3}\right) + 0 - \frac{1}{2}(-2) \\
&= -\left(-\frac{1}{12}\right) + 1 = \frac{13}{12}
\end{aligned}$$

It follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{13}{12} - \left(\frac{2}{3}\right)^2 = \frac{13}{12} - \frac{4}{9} = \frac{39}{36} - \frac{16}{36} = \frac{23}{36}$. ■

9. The discrete random variables X and Y are jointly distributed according to the given table:

$X \backslash Y$	1	2	3
-1	0.1	0.1	0.2
1	0.1	0	0.1
3	0.2	0.1	0.1

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\text{Cov}(X,Y)$ of X and Y. *[12]*

b. Let $U = -X - 2Y$. Compute $E(U)$ and $V(U)$. *[4]*

SOLUTIONS. **a.** We plug the numbers into the definitions and do much arithmetic:

$$\begin{aligned}
E(X) &= \sum_x xP(X = x) \\
&= (-1)(0.1 + 0.1 + 0.2) + 1(0.1 + 0 + 0.1) + 3(0.2 + 0.1 + 0.1) \\
&= -0.4 + 0.2 + 1.2 = 1 \\
E(Y) &= \sum_y yP(Y = y) \\
&= 1(0.1 + 0.1 + 0.2) + 2(0.1 + 0 + 0.1) + 3(0.2 + 0.1 + 0.1) \\
&= 0.4 + 0.4 + 1.2 = 2 \\
E\left(X^2\right) &= \sum_x x^2 P(X = x) \\
&= (-1)^2(0.1 + 0.1 + 0.2) + 1^2(0.1 + 0 + 0.1) + 3^2(0.2 + 0.1 + 0.1) \\
&= 0.4 + 0.2 + 3.6 = 4.2 \\
E\left(Y^2\right) &= \sum_y y^2 P(Y = y) \\
&= 1^2(0.1 + 0.1 + 0.2) + 2^2(0.1 + 0 + 0.1) + 3^2(0.2 + 0.1 + 0.1) \\
&= 0.4 + 0.8 + 3.6 = 4.8 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 4.2 - 1^2 = 3.2 \\
V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 4.8 - 2^2 = 0.8 \\
E(XY) &= \sum_{x,y} xyP(X = x \; \& \; Y = y) \\
&= (-1) \cdot 1 \cdot 0.1 + (-1) \cdot 2 \cdot 0.1 + (-1) \cdot 3 \cdot 0.2 \\
&\quad + 1 \cdot 1 \cdot 0.1 + 1 \cdot 2 \cdot 0 + 1 \cdot 3 \cdot 0.1 \\
&\quad + 3 \cdot 1 \cdot 0.2 + 3 \cdot 2 \cdot 0.1 + 3 \cdot 3 \cdot 0.1 \\
&= -0.1 - 0.2 - 0.6 + 0.1 + 0 + 0.3 + 0.6 + 0.6 + 0.9 = 1.6 \\
\text{Cov}(X,Y) &= E(XY) - E(X)E(Y) = 1.6 - 1 \cdot 2 = -0.4 \qquad \square
\end{aligned}$$

b. $E(U) = E(-X - 2Y) = -E(X) - 2E(Y) = -1 - 2 \cdot 2 = -5$ and

$$\begin{aligned} V(U) &= V(-X - 2Y) = V\left((-1)X + (-2)Y\right) \\ &= V\left((-1)X\right) + V\left((-2)Y\right) + 2\mathrm{Cov}\left((-1)X, (-2)Y\right) \\ &= (-1)^2 V(X) + (-2)^2 V(Y) + 2(-1)(-2)\mathrm{Cov}\left(X, Y\right) \\ &= 1 \cdot 3.2 + 4 \cdot 0.8 + 4 \cdot (-0.4) = 4.8\,. \qquad \blacksquare \end{aligned}$$

[Total = 100]

Part Die. Bonus!

- ⚀ . In series of games numbered 1, 2, 3, ... , the winning number in the nth game is randomly chosen from the set $\{1, 2, \ldots, n+2\}$. Kosh Naranek bets on the number 1 in each game and intends to keep playing until (s)he wins once. What is the probability that Kosh will have to play forever? *[1]*
- ⚅ . Write an original little poem about probability or mathematics in general. *[1]*

*[Part **Card** is on page 1.]*

Have a great August!