TRENT UNIVERSITY, FALL 2016

MATH 1550H Test

Monday, 11 July, 2016

Time: 50 minutes

Name: Solutions

STUDENT NUMBER: 3141592

Question	Mark
1	____
2	____
3	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *three (3)* of **a**–**d**. *[12 = 3 × 4 each]*

a. A fair coin is tossed ten times. What is the probability that at least two heads occur?

b. A hand of five cards is randomly drawn, without order or replacement, from a standard deck. What is the probability that you got exactly three of one kind and one of each of two other kinds in the hand?

c. Determine whether $f(t) = \begin{cases} 2e^{2t} & -\infty < t \leq 0 \\ 0 & \text{otherwise} \end{cases}$ is a probability density function.

d. A fair three-sided die with faces numbered 1 through 3 is rolled twice. What is the probability that the sum of the two rolls is even, given that the first roll was odd?

Solutions. **a.** Since the coin is tossed ten times, there are $2^{10} = 1024$ possible sequences of heads and tails as the outcomes; since the coin is fair, each sequence is equally likely and so has probability $m(\omega) = \frac{1}{2^{10}} = \frac{1}{1024}$. Note that $P(\text{at least 2 H}) = 1 - P(\text{0 H or 1 H})$. As $P(\text{0 H or 1 H}) = P(\text{0 H}) + P(\text{1 H})$ (note that you can't have 0 heads and 1 head at the same time), $P(\text{0 H}) = P(TTTTTTTTTT) = \frac{1}{2^{10}} = \frac{1}{1024}$, and $P(\text{1 H}) = \binom{10}{1}\frac{1}{2^{10}} = \frac{10}{1024}$, it follows that $P(\text{at least 2 H}) = 1 - \left[\frac{1}{1024} + \frac{10}{1024}\right] = 1 - \frac{11}{1024} = \frac{1013}{1024}$. □

b. There are $\binom{52}{5}$ possible hands; since each is equally likely, each has a probability of $\frac{1}{\binom{52}{5}}$ of being drawn. There are $\binom{13}{1}$ ways to pick the kind for the three-of-a-kind, $\binom{4}{3}$ ways to pick the three of that kind, $\binom{12}{2}$ ways to pick the kinds for the one-of-a-kinds, and $\binom{4}{1}$ ways each to pick one of those kinds. It follows that:

$$P(\text{3 of a kind \& 1 of each of 2 other kinds}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$$

$$= \frac{13 \cdot 4 \cdot 66 \cdot 4 \cdot 4}{2598960} = \frac{54912}{2598960} \approx 0.02113 \qquad \square$$

c. Since $e^x > 0$ for all $x \in \mathbb{R}$, $f(t) = 2e^{2t} \geq 0$ for all $t \leq 0$, and since $0 = 0$, $f(t) = 0 \geq 0$ for all $t > 0$, too. It remains to check that $\int_{-\infty}^{\infty} f(t)\, dt = 1$:

$$\int_{-\infty}^{\infty} f(t)\, dt = \int_{-\infty}^{0} 2e^{2t}\, dt + \int_{0}^{\infty} 0\, dt = \int_{-\infty}^{0} e^u\, du + 0 \qquad \text{Substituting } u = 2t, \text{ so } du = 2dt \text{ and } \begin{matrix} x & -\infty & 0 \\ u & -\infty & 0 \end{matrix}.$$

$$= e^u\big|_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1 \qquad \square$$

d. A three-sided die rolled once gives us three possible outcomes and, since the die is fair, each is equally likely and so has probability $\frac{1}{3}$. It folllows that $P(\text{1st roll odd}) = P(1) + P(3) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

A three-sided die rolled twice gives us $3 \cdot 3 = 9$ possible outcomes and, since the die is fair, each is equally likely and so has probability $\frac{1}{9}$. Out of the nine outcomes, four give an even sum for the two rolls after an odd number is rolled first: $\begin{matrix} + & 1 & 2 & 3 \\ 1 & \mathbf{2} & 3 & \mathbf{4} \\ 2 & 3 & 4 & 5 \\ 3 & \mathbf{4} & 5 & \mathbf{6} \end{matrix}$. It follows that $P(\text{1st roll odd \& sum even}) = 4 \cdot \frac{1}{9} = \frac{4}{9}$.

Thus $P(\text{sum even}|\text{1st roll odd}) = \frac{P(\text{1st roll odd \& sum even})}{P(\text{1st roll odd})} = \frac{4/9}{2/3} = \frac{4}{9} \cdot \frac{3}{2} = \frac{2}{3}$. ■

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. A baby's toy has four holes, numbered 1 through 4, and four balls, also numbered 1 through 4. If the baby randomly puts a ball into each hole, what is the probability that at least one ball ends up in a hole with the same number?

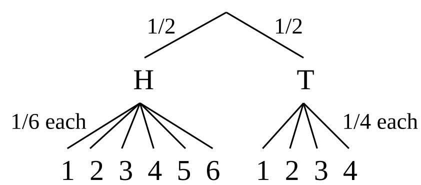
b. A fair coin is tossed: if it come up heads, a fair standard die is rolled once, but if the coin comes up tails, a fair four-sided die with faces numbered 1 through 4 is rolled once instead. Draw the complete tree diagram for this experiment and determine the probability that the die roll gives a number that is at least 3.

c. Suppose A and B are independent events in a sample space Ω. Verify that A and $\overline{B}$ are also independent.

SOLUTIONS. **a.** There are $4! = 24$ ways to stuff one of the four balls into each of the four holes. each is equally likely here, so each has a probability of $\frac{1}{4!}$. For each of the $\binom{4}{1} = 4$ ways to put a ball into the hole with the same number, there are $3! = 6$ ways to distribute the other balls among the other three holes. However, this overcounts the arrangements where two or more balls are put in the correctly numbered hole, so we need to subtract the $\binom{4}{2} = 6$ ways to put two balls in the correct hole times the $2! = 2$ ways to arrange the other two balls. However, this last takes away the arrangements where three or more balls are put in the correctly numbered hole, so we need to add back the $\binom{4}{3} = 4$ ways to put three balls in the correct hole times the $1! = 1$ ways to arrange the other two balls. Finally, the last overcounts the arrangements where all four balls end up in the correct hole too many times, so we need to take away the $\binom{4}{4} = 1$ way to put all four balls in the right holes times the $0! = 1$ way of arranging the remaining balls. Thus the desired probability is:

$$\begin{aligned} P(\text{at least one match}) &= \frac{\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!}{4!} \\ &= \frac{4\cdot 6 - 6\cdot 2 + 4\cdot 1 - 1\cdot 1}{24} = \frac{24 - 12 + 4 - 1}{24} = \frac{15}{24} = \frac{5}{8} \qquad \square \end{aligned}$$

b. Here's the tree:



Using the information in the tree:

$$\begin{aligned} P(\text{die roll was at least 3}) &= m(H3) + m(H4) + m(H5) + m(H6) + m(T3) + m(T4) \\ &= 4\cdot\frac{1}{2}\cdot\frac{1}{6} + 2\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \qquad \square \end{aligned}$$

c. Since $B \cap \overline{B} = \emptyset$, we also have that $(A \cap B) \cap \left(A \cap \overline{B}\right) = \emptyset$, and since $(A \cap B) \cup \left(A \cap \overline{B}\right) = A \cap \left(B \cup \overline{B}\right) = A \cap \Omega = A$, it follows that $P(A) = P(A \cap B) + P\left(A \cap \overline{B}\right)$. Rearranging this last, and using the fact that A and B are independent, so $P(A \cap B) = P(A)P(B)$, gives us that:

$$P\left(A \cap \overline{B}\right) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)\left[1 - P(B)\right] = P(A)P\left(\overline{B}\right)$$

Thus A and $\overline{B}$ are independent. ■

3. Do any *one (1)* of **a** or **b**. *[8 = 1 × 8 each]*

a. Given the density function $g(t) = \begin{cases} t^{-2} & 1 \leq t < \infty \\ 0 & t < 1 \end{cases}$, let $A = [0, 2]$ and $B = [1, 3]$ be events. Compute $P(A|B)$.

b. A five-card hand is randomly drawn, without order or replacement, from a standard deck. What is the probability that at least three of the cards in the hand are from the same suit?

Solutions. **a.** First, using the Power Rule for integration:

$$P(B) = \int_1^3 g(t)\,dt = \int_1^3 t^{-2}\,dt = \left.\frac{t^{-2+1}}{-2+1}\right|_1^3 = \left.-t^{-1}\right|_1^3 = \left.-\frac{1}{t}\right|_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$$

Second, note that $A \cap B = [1, 2]$. Then, again using the Power Rule:

$$P(A \cap B) = \int_1^2 g(t)\,dt = \int_1^2 t^{-2}\,dt = \left.\frac{t^{-2+1}}{-2+1}\right|_1^2 = \left.-t^{-1}\right|_1^2 = \left.-\frac{1}{t}\right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

It follows that $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/2}{2/3} = \dfrac{1}{2} \cdot \dfrac{3}{2} = \dfrac{3}{4}$. □

b. Recall that there are four suits of thirteen kinds each, and that the number of five-card hands that can be drawn from a standard deck without order or replacement is $\binom{52}{5}$, so each has a probability of $1/\binom{52}{5}$ of coming up in a random draw.

$$\begin{aligned}
P(\geq 3 \text{ of same suit}) &= P(3 \text{ of one suit \& 2 of another}) \\
&\quad + P(3 \text{ of one suit \& 1 of each of two other suits}) \\
&\quad + P(4 \text{ of one suit \& 1 of another suit}) + P(5 \text{ of one suit}) \\
&= \frac{\binom{4}{1}\binom{13}{3}\binom{3}{1}\binom{13}{2}}{\binom{52}{5}} + \frac{\binom{4}{1}\binom{13}{3}\binom{3}{2}\binom{13}{1}\binom{13}{1}}{\binom{52}{5}} + \frac{\binom{4}{1}\binom{13}{4}\binom{3}{1}\binom{13}{1}}{\binom{52}{5}} + \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} \\
&= \frac{4 \cdot 286 \cdot 3 \cdot 78}{2598960} + \frac{4 \cdot 286 \cdot 3 \cdot 13 \cdot 13}{2598960} + \frac{4 \cdot 715 \cdot 3 \cdot 13}{2598960} + \frac{4 \cdot 1287}{2598960} \\
&= \frac{267696 + 580008 + 111540 + 5148}{2598960} = \frac{964392}{2598960} \approx 0.37107
\end{aligned}$$

The simplification above is pretty much gratuitous, unless you happen to need it. :-) □

[Total = 30]