Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2016

Solutions to the Quizzes

Quiz #1. Wednesday, 22 June, 2016. *[10 minutes]*

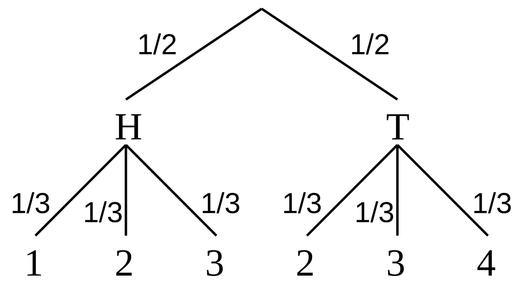
A fair coin is tossed. If the coin comes up heads, a fair three-sided die with faces labelled 1, 2, and 3, respectively, is rolled once; if the coin comes up tails, a fair three-sided die with faces labelled 2, 3, and 4, respectively, is rolled once.

1. What are the sample space and the probability mass function for this experiment? *[2]*
2. Draw the complete tree diagram for this experiment. *[1.5]*
3. What is the probability that the die roll in the second stage of the experiment produced a 2? *[1.5]*

SOLUTIONS. 1. The sample space is $\Omega = \{ (H,1), (H,2), (H,3), (T,2), (T,3), (T,4) \}$. The probability mass function turns out to be uniform: for each outcome we toss the fair coin, with $P(H) = P(T) = \frac{1}{2}$, and then roll a fair three-sided die, with each face having a probability of $\frac{1}{3}$, so each outcome $\omega \in \Omega$ has probability $m(\omega) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. That is,

$$m(H,1) = m(H,2), = m(H,3) = m(T,2) = m(T,3) = m(T,4) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}. \quad \square$$

2.



$\square$

3. There are two outcomes that have a 2 as the face that occurred in the roll of the die, namely $(H,2)$ and $(T,2)$. Thus the probability of this event, call it A, is

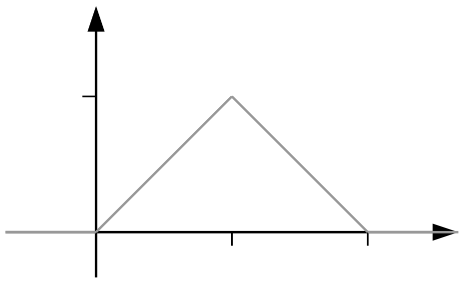
$$P(A) = m(H,2) + m(T,2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}. \quad \blacksquare$$

Quiz #2. Monday, 27 June, 2016. *[10 minutes]*

Consider the function $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$.

0. Sketch the graph of $f(t)$. *[1]*
1. Verify that $f(t)$ is a probability density function. *[2]*
2. Compute $P\left(\left[\frac{1}{2}, \infty\right)\right)$. *[2]*

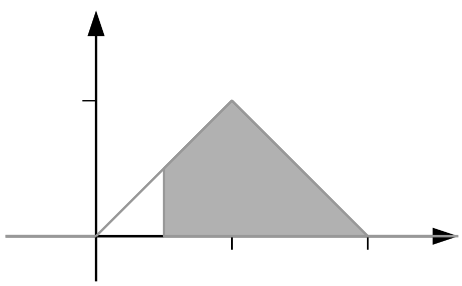
SOLUTIONS. 1.



$\square$

2. As may be seen from the graph, $f(t)$ is continuous and ≥ 0 everywhere. Since the graph is 0 everywhere else, the only part contributing any area under the graph is the part between 0 and 2. This part of the graph is a triangle with base 2 and height 1, so the area under the graph of $f(t)$ is $\frac{1}{2} \cdot 2 \cdot 1 = 1$, as required. Thus $f(t)$ is indeed a probability density function. $\square$

3. $P\left(\left[\frac{1}{2}, \infty\right)\right)$ is the area under the graph of $f(t)$ from $t = \frac{1}{2}$ onward:



This area is not too hard to compute directly by, say, breaking it up into triangular pieces, but it is even easier to observe that the only area under the graph that is not included is a trangle with base $\frac{1}{2}$ and height $\frac{1}{2}$. It follows that $P\left(\left[\frac{1}{2}, \infty\right)\right) = 1 - P\left(\left(-\infty, \frac{1}{2}\right)\right) = 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} - 1 - \frac{1}{8} = \frac{7}{8}$. $\blacksquare$

Quiz #3. Wednesday, 29 June, 2016. *[10 minutes]*

A five-card hand is randomly drawn, all at once and without replacement, from a standard 52-card deck.

1. What is the probability that the hand is a *full house*, consisting of three of one kind and two of another kind? *[5]*

SOLUTION. There are $\binom{52}{5}$ ways to pick a five-card hand, without order and without replacement, from a standard deck of 52 cards. In a random draw, each possible hand is equally likely, so any particular hand has a probability of $1/\binom{52}{5}$ of being drawn.

To count the number of possible hands that are a full house, note first that there are $\binom{13}{1}$ ways to pick the suit of the three of a kind, and then $\binom{4}{3}$ ways to pick the three of that kind. (Recall that there are four of each kind in the deck.) After this, there are $\binom{12}{1}$ ways to pick a kind for the pair from the twelve remaining kinds, and then $\binom{4}{3}$ ways to pick the two of that kind that make the pair. It follows that there are $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ possible full house hands.

Combining the stuff above, we get:

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{13 \cdot 4 \cdot 12 \cdot 6}{2598960} = \frac{3744}{2598960} \approx 0.00144$$

I'd be happy if you left it as $\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$ ... ■

Quiz #4. Monday, 4 July, 2016. *[10 minutes]*

1. The four cards $A\heartsuit$, $A\diamondsuit$, $A\clubsuit$, and $A\spadesuit$ are laid out for you in that order. You shuffle the four cards and lay them out again in a random order. What is the probability that none of the four cards ends up in the same position it was in when the hand was originally given to you? *[5]*

SOLUTION. We will compute the probability that at least one card ends up in the same position and subtract that from 1. This problem is basically just like the hat-check problem, the poster child for the inclusion-exclusion principle. Note that there are $4! = 24$ ways to lay the four cards in some order; with a random choice, each layout is equally likely and so has a probability of $\frac{1}{4!} = \frac{1}{24}$.

There are $\binom{4}{1} = 4$ ways to pick a card that will end up in the same position, and $3! = 6$ ways to lay out the remaining three cards, for a probability of $\binom{4}{1}\frac{3!}{4!} = 4 \cdot \frac{6}{24} = 1$. However, this overcounts the layouts where at least two cards end up in the same spot, so we have to take away the probability of that occurrence.

There are $\binom{4}{2} = 6$ ways to pick two cards that that will end up in the same position, and $2! = 2$ ways to lay out the remaining two cards, for a probability of $\binom{4}{2}\frac{2!}{4!} = 6 \cdot \frac{2}{24} = \frac{1}{2}$. However, this, in turn, overcounts the cases where at least three (and hence all four) cards end up in the same spot, so, since we are taking the probability we just computed away, we have to add the overcounted probability back.

There are $\binom{4}{3} = 4$ ways to pick three cards that will end up in the same position, and $1! = 1$ way to lay out the remaining card, for a probability of $\binom{4}{3}\frac{1!}{4!} = 4 \cdot \frac{1}{24} = \frac{1}{6}$. However, this analysis, in turn, overcounts the cases where all four cards end up in the same spot, so, since we are adding in the probability we just computed, we have to take away the overcounted probability.

Continuing in the same style, there is $\binom{4}{4} = 1$ way to pick all four cards to end up in the same position, and $0! = 1$ way to lay out the remaining cards. for a probability of $\binom{4}{4}\frac{0!}{4!} = 1 \cdot \frac{1}{24} = \frac{1}{24}$.

It follows that

$$\begin{aligned} P(\geq 1 \text{ card in the same position}) &= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{24}{24} - \frac{12}{24} + \frac{4}{24} - \frac{1}{24} \\ &= \frac{24 - 12 + 4 - 1}{24} = \frac{15}{24} = \frac{5}{8}, \end{aligned}$$

so

$$P(\text{no card in the same position}) = 1 - P(\geq 1 \text{ card in the same position} = 1 - \frac{5}{8} = \frac{3}{8}.$$

Whew! ■

Quiz #5. Wednesday, 6 July, 2016. *[10 minutes]*

A fair coin is tossed four times. Let A be the event that there were at least three heads, and let B be the event that at least one of the first two tosses was a head.

1. Determine whether A and B are independent or not. *[3]*
2. Compute $P(A|B)$. *[2]*

SOLUTIONS. 1. Recall that A and B are independent exactly when $P(A \cap B) = P(A)P(B)$, so we will first compute $P(A)$, $P(B)$, and $P(A \cap B)$. Although we could do so using counting techniques, we'll use brute force. The sample space is

$$\begin{aligned}\Omega = \{\, & HHHH,\ HHHT,\ HHTH,\ HTHH,\ THHH,\ HHTT,\ HTHT,\ HTTH \\ & THHT,\ THTH,\ TTHH,\ HTTT,\ THTT,\ TTHT,\ TTTH,\ TTTT\,\}\end{aligned}$$

and, since the coin is fair, every one of the sixteen outcomes is equally likely, so $m(\omega) = \dfrac{1}{16}$ for each outcome $\omega \in \Omega$. It follows that:

$$\begin{aligned}
P(A) &= P\,(\,HHHH,\ HHHT,\ HHTH,\ HTHH,\ THHH\,) = 5 \cdot \frac{1}{16} = \frac{5}{16} \\
P(B) &= P\,(\,HHHH,\ HHHT,\ HHTH,\ HTHH,\ THHH,\ HHTT, \\
&\qquad HTHT,\ HTTH,\ THHT,\ THTH,\ HTTT,\ THTT\,) = 12 \cdot \frac{1}{16} = \frac{3}{4} \\
P(A \cap B) = P(A) &= P\,(\,HHHH,\ HHHT,\ HHTH,\ HTHH,\ THHH\,) = 5 \cdot \frac{1}{16} = \frac{5}{16}
\end{aligned}$$

Note that $A \subseteq B$, so $A \cap B = A$.

Since $P(A \cap B) = \dfrac{5}{16} \neq \dfrac{15}{64} = \dfrac{5}{16} \cdot \dfrac{3}{4} = P(A)P(B)$, A and B are not independent. $\Box$.

2. Using the definition and the information computed above,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{5/16}{3/4} = \frac{5}{16} \cdot \frac{4}{3} = \frac{5}{12}\,. \qquad \blacksquare$$

Quiz #6. Wednesday, 13 July, 2016. *[10 minutes]*

An otherwise fair seven-sided die has single faces labelled 1, 2, 4, and 5, respectively, and three faces labelled 3. The die is rolled once; let X be the number that comes up on the roll.

1. Compute the expected value, $E(X)$, of X. *[2.5]*
2. Compute the variance, $V(X)$, of X. *[2.5]*

SOLUTIONS. 1. Note that $P(X=1) = P(X=2) = P(X=4) = P(X=5) = \frac{1}{7}$ and $P(X=3) = \frac{3}{7}$. Applying the definition of expected value yields:

$$\begin{aligned} E(X) &= 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + 4 \cdot P(X=4) + 5 \cdot P(X=5) \\ &= 1 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7} + 3 \cdot \frac{3}{7} + 4 \cdot \frac{1}{7} + 5 \cdot \frac{1}{7} = (1+2+9+4+5) \cdot \frac{1}{7} = 21 \cdot \frac{1}{7} = 3 \quad \square \end{aligned}$$

2. We will use the formula $V(X) = E\left(X^2\right) - [E(X)]^2$ to compute the variance of X. We have $E(X) = 3$ form the solution to question 1, but we still need to compute $E\left(X^2\right)$. Applying the definition of expected value to X^2 yields:

$$\begin{aligned} E\left(X^2\right) &= 1^2 \cdot P(X=1) + 2^2 \cdot P(X=2) + 3^2 \cdot P(X=3) \\ &\quad + 4^2 \cdot P(X=4) + 5^2 \cdot P(X=5) \\ &= 1 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 9 \cdot \frac{3}{7} + 16 \cdot \frac{1}{7} + 25 \cdot \frac{1}{7} \\ &= (1+4+27+16+25) \cdot \frac{1}{7} = 73 \cdot \frac{1}{7} = \frac{73}{7} \end{aligned}$$

It follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{73}{7} - 3^2 = \frac{73}{7} - 9 = \frac{73}{7} - \frac{63}{7} = \frac{10}{7}$. ■

Quiz #7. Monday, 18 July, 2016. *[10 minutes]*

Suppose that the continuous random variable X has the probability density function

$$f(t) = \begin{cases} \frac{3}{4}\left(1-t^2\right) & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} .$$

1. Compute the expected value, $E(X)$, of X. *[2.5]*
2. Compute the variance, $V(X)$, of X. *[2.5]*

Solutions. 1. By definition, with some help from the Power Rule for integration:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} tf(t)\,dt = \int_{-\infty}^{-1} t \cdot 0\,dt + \int_{-1}^{1} t \cdot \frac{3}{4}\left(1-t^2\right)\,dt + \int_{1}^{\infty} t \cdot 0\,dt \\ &= 0 + \int_{-1}^{1} \left(t - t^3\right)\,dt + 0 = \frac{3}{4}\left(\frac{t^2}{2} - \frac{t^4}{4}\right)\Big|_{-1}^{1} \\ &= \frac{3}{4}\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \frac{3}{4}\left(\frac{(-1)^2}{2} - \frac{() - 1^4}{4}\right) = \frac{3}{4} \cdot \frac{1}{4} - \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16} - \frac{3}{16} = 0 \quad \square \end{aligned}$$

2. First, we compute $E\left(X^2\right)$:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} t^2 f(t)\,dt = \int_{-\infty}^{-1} t^2 \cdot 0\,dt + \int_{-1}^{1} t^2 \cdot \frac{3}{4}\left(1-t^2\right)\,dt + \int_{1}^{\infty} t^2 \cdot 0\,dt \\ &= 0 + \int_{-1}^{1} \left(t^2 - t^4\right)\,dt = \frac{3}{4}\left(\frac{t^3}{3} - \frac{t^5}{5}\right)\Big|_{-1}^{1} \\ &= \frac{3}{4}\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \frac{3}{4}\left(\frac{(-1)^3}{3} - \frac{(-1)^5}{5}\right) = \frac{3}{4} \cdot \frac{2}{15} - \frac{3}{4} \cdot \frac{-2}{15} = \frac{3}{4} \cdot \frac{4}{15} = \frac{3}{15} = \frac{1}{5} \end{aligned}$$

It then follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{1}{5} - 0^2 = \frac{1}{5}$. $\blacksquare$

Quiz #8. Wednesday, 20 July, 2016. *[10 minutes]*

Suppose the continuous random variable X has a normal distribution with expected value $\mu = 10$ and standard deviation $\sigma = 5$.

1. Compute $P(7 \leq X \leq 17)$. *[3]*
2. Find the value of c for which $P(X \leq c) \approx 0.7734$. *[2]*

SOLUTIONS. 1. Recall that if X has a normal distribution with $\mu = 10$ and $\sigma = 5$, then $Z = \frac{X-10}{5}$ has a standard normal distribution, for which we have a table.

$$\begin{aligned} P(7 \leq X \leq 17) &= P(-3 \leq X - 10 \leq 7) = P\left(-\frac{3}{5} \leq \frac{X-10}{5} \leq \frac{7}{5}\right) \\ &= P(-0.6 \leq Z \leq 1.4) = P(Z \leq 1.4) - P(Z \leq -0.6) \\ &\approx 0.9192 - 0.2743 = 0.6449 \quad \square \end{aligned}$$

2. As in the solution to question 1, we convert the problem into one about $Z = \frac{X-10}{5}$ and use the standard normal table.

$$P(X \leq c) = P\left(X - 10 \leq c - 10\right) = P\left(\frac{X-10}{5} \leq \frac{c-10}{5}\right) = P\left(Z \leq \frac{c-10}{5}\right)$$

Consulting our table, we see that $P(Z \leq a) \approx 0.7734$ when $a = 0.75$. Thus we must have $\frac{c-10}{5} = 0.75$, so $c = 5 \cdot 0.75 + 10 = 13.75$. ■

Quiz #9. Monday, 25 July, 2016. *[however many minutes]*

Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	-1	2
2	0.1	0.1
3	0.2	0.1
5	0.3	0.2

1. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\mathrm{Cov}(X,Y)$ of X and Y. *[5]*

SOLUTION. Here goes, with adding up the probabilities in a given row or column of the table done in one's head:

$$\begin{aligned} E(X) &= 2 \cdot 0.2 + 3 \cdot 0.3 + 5 \cdot 0.5 = 0.4 + 0.9 + 2.5 = 3.8 \\ E(Y) &= (-1) \cdot 0.6 + 2 \cdot 0.4 = -0.6 + 0.8 = 0.2 \\ E\left(X^2\right) &= 2^2 \cdot 0.2 + 3^2 \cdot 0.3 + 5^2 \cdot 0.5 = 0.8 + 2.7 + 12.5 = 16 \\ E\left(Y^2\right) &= (-1)^2 \cdot 0.6 + 2^2 \cdot 0.4 = 0.6 + 1.6 = 2.2 \\ V(X) &= E\left(X^2\right) - [E(X)]^2 = 16 - 3.8^2 = 16 - 14.44 = 1.56 \\ V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 2.2 - 0.2^2 = 2.2 - 0.04 = 2.16 \\ E(XY) &= 2 \cdot (-1) \cdot 0.1 + 2 \cdot 2 \cdot 0.1 + 3 \cdot (-1) \cdot 0.2 + 3 \cdot 2 \cdot 0.1 + 5 \cdot (-1) \cdot 0.3 + 5 \cdot 2 \cdot 0.2 \\ &= -0.2 + 0.4 - 0.6 + 0.6 - 1.5 + 2.0 = 0.7 \\ \mathrm{Cov}(X,Y) &= E(XY) - E(X) \cdot E(Y) = 0.7 - 3.8 \cdot 0.2 = 0.7 - 0.76 = -0.06 \quad \blacksquare \end{aligned}$$

Quiz #10. Wednesday, 27 July, 2016. *[10 minutes]*

Suppose X is a random variable with $X \geq 0$, expected value $E(X) = 10$, and variance $V(X) = 4$.

1. What can you say about $P(X < 20)$ with the help of Markov's Inequality? *[2]*
2. What can you say about $P(X < 20)$ with the help of Chebyshev's Inequality? *[3]*

SOLUTIONS. 1. As a sanity check, note that we can use Markov's Inequality because $X \geq 0$. Markov's Inequality tells us that

$$P(X \geq 20) \leq \frac{E(X)}{20} = \frac{10}{20} = \frac{1}{2},$$

so

$$P(X < 20) = 1 - P(X \geq 20) \geq 1 - \frac{1}{2} = \frac{1}{2}. \quad \square$$

2. As in the solution above, note that $P(X < 20) = 1 - P(X \geq 20)$. Since

$$\begin{aligned} P(X \geq 20) &= P(X - 10 \geq 20 - 10) \\ &= P(X - 10 \geq 10) \\ &\leq P\left(|X - 10| \geq 10\right) \\ &\leq \frac{V(X)}{10^2} = \frac{4}{100} = \frac{1}{25}, \end{aligned}$$

using Chebyshev's Inequality towards the end, it follows that

$$P(X < 20) = 1 - P(X \geq 20) \geq 1 - \frac{1}{25} = \frac{24}{25}. \quad \blacksquare$$