Mathematics 1550H – Introduction to probability
Trent University, Summer 2016

Final Examination
Wednesday, 3 July, 2016

Time-space: 14:00–17:00 in GCS 110 *Inflicted by* Стефан Біланюк.

Instructions: Do both of parts **X** and **Y**, and, if you wish, part **Z**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one 8.5″ × 11″ or A4 aid sheet; standard normal table; one brain maximum.

Part X. Do all of **1**–**5**. *[Subtotal = 68/100]*

1. A fair coin is tossed three times. Let A be the event that there are at least two heads in the three tosses and let B be the event that there are exactly two heads among the three tosses.

a. Draw the complete tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Compute $P(A)$, $P(B)$, $P(A|B)$, and $P(B|A)$. *[7]*

2. Let U be a continuous random variable with the following probability density function:

$$g(t) = \begin{cases} 1+t & -1 \leq t \leq 0 \\ 1-t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $g(t)$ is indeed a probability density function. *[5]*

b. Compute the expected value, $E(U)$, and variance, $V(U)$, of U. *[10]*

3. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck. Let A be the event that the hand includes two pairs (of different kinds, with the fifth card of a third kind), and let B be the event that exactly three kinds occur in the hand. Compute the conditional probabilities $P(A|B)$ and $P(B|A)$. *[10]*

4. Two fair standard dice are rolled simultaneously, with the experiment being repeated until both dice come up even on the same roll. Let Y be the number of rolls required, and let A be the event that the sum of the faces of the two dice on the last roll was eight.

a. Compute the expected value, $E(Y)$, and variance, $V(Y)$, of Y. *[7]*

b. Compute $P(A)$. *[6]*

5. Suppose X is a continuous random variable that has a normal distribution with expected value $\mu = 8$ and variance $\sigma^2 = 4$.

a. Compute $P(X \geq 12)$ with the help of a standard normal table. *[5]*

b. What does Markov's Inequality tell you about $P(X \geq 12)$? *[5]*

c. What can Chebyshev's Inequality help tell you about $P(X \geq 12)$? *[5]*

*[Parts **Y** and **Z** are on page 2.]*

Part Y. Do any *two* (2) of **6–9**. *[Subtotal = 32/100]*

6. Let $g(t) = \begin{cases} 2te^{-t^2} & t \geq 0 \\ 0 & t < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(t)$ is indeed a probability density function. *[8]*

b. Find the *median* of X, *i.e.* the number m such that $P(X \leq m) = \frac{1}{2} = 0.5$. *[8]*

7. A box contains six balls, two red and four green. Balls are drawn randomly from the box, with replacement between draws, until the second time a red ball is drawn. Let X be the number of draws made during the process.

a. What is the probability function of X? *[5]*

b. What are the expected value, $E(X)$, and variance, $V(X)$, of X? *[5]*

c. Let A be the event that $X = 3$, and let B be the event that the first ball drawn is red. Determine whether A and B are independent events or not. *[6]*

8. For each $i = 1, 2, \ldots, 10$, X_i is a random variable that gives 0 or 1 if the ith toss of a fair coin came up T or H, respectively. Let $X = X_1 + X_2 + \cdots + X_{10}$.

a. Compute the expected value, $E(X)$, and variance, $V(X)$, of X. *[6]*

b. What is the probability function of X? *[10]*

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	-1	0	1
1	0.1	0.1	0.1
3	0	0.2	0.1
4	0.2	0.1	0.1

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\text{Cov}(X, Y)$ of X and Y. *[12]*

b. Let $W = X - Y$. Compute $E(W)$ and $V(W)$. *[4]*

[Total = 100]

Part Z. Bonus!

○●. There are 64 teams who play in a single elimination tournament (hence 6 rounds), and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points. *[1]*

●○. Write an original little poem about probability or mathematics in general. *[1]*

*[Part **X** is on page 1.]*

I HOPE THAT YOU ENJOYED THE COURSE.
ENJOY THE SUMMER!