Trent University, Summer 2015

MATH 1550H Test

Monday, 11 July, 2015

Time: 50 minutes

Name: Solutions

Student Number: 0123456

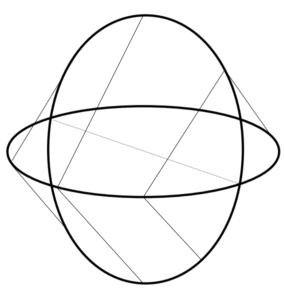
Question	Mark
1	____
2	____
3	____
Bonus	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

Bonus: For each $n \geq 1$, describe a shape that would give you a fair n-sided die. *[1]*

Solution. $n = 1$: sphere; $n = 2$: disk; $n \geq 3$: take a sphere, mark n equally spaced lines of longitude from pole to pole, shave away excess material by running a cutting edge from pole to pole along each pair of neighbouring marked lines of longitude. Here is a very crude sketch of the shape for $n = 4$:



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1. Do any *three (3)* of **a**–**e**. *[12 = 3 × 4 each]*

a. A fair trihedral (3-sided) die (with faces numbered 1, 2, and 3, respectively) is rolled twice. Let Y be the sum of the two rolls. Compute the probability that Y is even.

b. How many 3-letter sequences can be formed using the letters in the word "pastrami" if the two "a"s cannot be distinguished?

c. Five cards are simultaneously drawn at random from a standard 52-card deck. What is the probability of drawing a full house (3 of a kind plus 2 of a kind)?

d. A fair coin is tossed three times, and let X be the number of heads minus the number of tails. Find the probability function of X.

e. Suppose the continuous random variable W has a normal distribution with $\mu = 2$ and $\sigma = 3$. Compute $P(W \leq 5)$.

SOLUTIONS. **a.** Here is a table giving the sums of the possible rolls:

+	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

Note that there are $3 \cdot 3 = 9$ possible, equally likely, pairs of rolls. Using this fact and the table, we have:

$$P(Y \text{ is even}) = P(1,1) + P(1,3) + P(2,2) + P(3,1) + P(3,3) = 5 \cdot \frac{1}{9} = \frac{5}{9} \qquad \square$$

b. Counting the two "a"s, there are eight letters in "pastrami." A 3-letter sequence using these letters could include two, one, or none of the "a"s, and we will count the number of possible sequences in each case separately.

First, if the 3-letter sequence includes both "a"s, it can include only one of the remaining six letters. There are $\binom{6}{1} = 6$ ways to choose the remaining letter and $\binom{3}{1} = 3$ ways to pick the position in the sequence that it occupies, for a total of $6 \cdot 3 = 18$ sequences that include both "a"s.

Second, if the 3-letter sequence includes just one "a," it must include two of the remaining six letters. There are $\binom{6}{2} = 15$ ways to choose these two letters, $\binom{3}{2} = 3$ ways to choose the two slots they occupy, and $2! = 2$ ways to arrange them in these slots, for a total of $15 \cdot 3 \cdot 2 = 90$ sequences that include one "a."

Third, if the 3-letter sequence includes neither "a," it must include three of the remaining six letter. There are $\binom{6}{3} = 20$ ways to pick the three letters and $3! = 6$ ways to arrange them in the three slots, for a total of $20 \cdot 6 = 120$ sequences that include no "a."

This gives a total of $18 + 90 + 120 = 228$ sequences with 3-letters that can be formed using the letters in the word "pastrami" if the two "a"s cannot be distinguished. $\square$

c. Recall that in a standard deck there are thirteen kinds and four of each kind, one for each suit, for a total of 52 cards. The five cards in the hand are not drawn in order, so there are a total of $\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = 2598960$ equally likely possible hands. For a full house, with three of one kind and two of another kind, there are $\binom{13}{1} = 13$ ways to pick the kind for the three and $\binom{4}{3} = 4$ ways to pick the three out of four of that kind, leaving $\binom{12}{1} = 12$ ways to pick the kind for the two and $\binom{4}{2} = 6$ ways to pick the two out of four of that kind. It follows that the probability of drawing a full house is:

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{13 \cdot 4 \cdot 12 \cdot 6}{2598960} = \left(\frac{3744}{2598960}\right) = \frac{6}{4165} \approx 0.0014 \quad \square$$

d. The possible outcomes for X are $0 - 3 = -3$ (*i.e.* TTT), $1 - 2 = -1$ (*i.e.* HTT, THT, or TTH), $2 - 1 = 1$ (*i.e.* HHT, HTH, or THH), and $3 - 0 = 3$ (*i.e.* HHH). Note that since the coin is fair, any sequence of three heads and/or tails has probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Thus the probability function of X is given by:

$$\begin{aligned}
p(-3) &= P(X = -3) = P(TTT) = \frac{1}{8} \\
p(-1) &= P(X = -1) = P(HTT) + P(THT) + P(TTH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \\
p(1) &= P(X = 1) = P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \\
p(3) &= P(X = 3) = P(HHH) = \frac{1}{8}
\end{aligned}$$

Of course, $p(n) = 0$ if $n \neq -3, -1, 1$, or 3. $\square$

e. If W has a normal distribution with $\mu = 2$ and $\sigma = 3$, then $Z = \frac{W-2}{3}$ has a standard normal distribution. Note that $W \leq 5$ exactly when $Z = \frac{W-2}{3} \leq \frac{5-2}{3} = \frac{3}{3} = 1$, so $P(W \leq 5) = P(Z \leq 1)$. Looking up $P(Z \leq 1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-x^2/2}\, dx$ on the cumulative standard normal table then tells us that $P(W \leq 5) = P(Z \leq 1) \approx 0.8413$. ■

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Compute $P(A|B) + P(\bar{A}|B)$, where A and B are events in a sample space Ω.

b. A fair die is rolled twice. Let A be the event that the two rolls give a different number and B be the event that the sum of the two rolls is even. Determine whether A and B are independent or not.

c. Let $f(x) = \begin{cases} x^{-2} & x \geq 1 \\ 0 & x < 1 \end{cases}$ be the probability density function of the continuous random varable X. Let A be the event that $X \leq 2$ and let B be the event that $0 \leq X \leq 3$. Compute $P(A|B)$.

SOLUTIONS. **a.** Note that since $A \cap \bar{A} = \emptyset$, $(A \cap B) \cap (\bar{A} \cap B) = \emptyset$ too, and so $P\left((A \cap B) \cup (\bar{A} \cap B)\right) = P(A \cap B) + P(\bar{A} \cap B) - P(\emptyset) = P(A \cap B) + P(\bar{A} \cap B)$ as $P(\emptyset) = 0$. As we also have that $A \cup \bar{A} = \Omega$, it follows that

$$\begin{aligned} P(A|B) + P(\bar{A}|B) &= \frac{P(A \cap B)}{P(B)} + \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(A \cap B) + P(\bar{A} \cap B)}{P(B)} \\ &= \frac{P\left((A \cap B) \cup (\bar{A} \cap B)\right)}{P(B)} = \frac{P\left((A \cup \bar{A}) \cap B\right)}{P(B)} \\ &= \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1\,. \qquad \square \end{aligned}$$

b. We will take the brute-force approach. First, here is a table of the possible – and equally likely! – outcomes for the two rolls and whether the sum is odd (o) or even (e).

+	1	2	3	4	5	6
1	e	o	e	o	e	o
2	o	e	o	e	o	e
3	e	o	e	o	e	o
4	o	e	o	e	o	e
5	e	o	e	o	e	o
6	o	e	o	e	o	e

Note that there are $6 \cdot 6 = 36$ possible outcomes for the two rolls, 18 of which give an even sum and 18 or which give an odd sum.

A consists of the outcomes where the second roll is not the same as the first, *i.e.* the 30 outcomes not on the diagonal in the table. Thus $P(A) = \frac{30}{36} = \frac{5}{6}$.

B consists of all the outcomes for which the sum is even, so $P(B) = \frac{18}{36} = \frac{1}{2}$.

$A \cap B$ consists of all the outcomes not on the diagonal for which the sum is even. Since all six entries on the diagonal are e, there are $18 - 6 = 12$ outcomes in $A \cap B$, so $P(A \cap B) = \frac{12}{36} = \frac{1}{3}$.

As $P(A \cap B) = \frac{1}{3} \neq \frac{5}{12} = \frac{5}{6} \cdot \frac{1}{2} = P(A)P(B)$, the A and B are not independent. $\square$

c. $A = (-\infty, 2]$ and $B = [0, 3]$, so $A \cap B = [0, 2]$. By definition, we have

$$\begin{aligned} P(B) &= \int_0^3 f(x)\,dx = \int_0^1 0\,dx + \int_1^3 x^{-2}\,dx = 0 + \left.\frac{x^{-2+1}}{-2+1}\right|_1^3 \\ &= -x^{-1}\Big|_1^3 = \left.\frac{-1}{x}\right|_1^3 = \frac{-1}{3} - \frac{-1}{1} = 1 - \frac{1}{3} = \frac{2}{3} \\ \text{and} \quad P(A \cap B) &= \int_0^2 f(x)\,dx = \int_0^1 0\,dx + \int_1^2 x^{-2}\,dx = 0 + \left.\frac{x^{-2+1}}{-2+1}\right|_1^2 \\ &= -x^{-1}\Big|_1^2 = \left.\frac{-1}{x}\right|_1^2 = \frac{-1}{2} - \frac{-1}{1} = 1 - \frac{1}{2} = \frac{1}{2}\,. \end{aligned}$$

It follows that $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{2}}{\frac{2}{3}} = \dfrac{1}{2} \cdot \dfrac{3}{2} = \dfrac{3}{4}$. ■

3. Do any *one (1)* of **a** or **b**. *[8 = 1 × 8 each]*

a. Suppose Z is a continuous random variable with an exponential distribution, so it has density function $h(z) = \begin{cases} \lambda e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$ for some $\lambda > 0$, and suppose $P(Z \leq 2) = \frac{1}{2}$. Determine λ.

b. A bin contains five blue and five red balls. If balls are drawn randomly, without replacement, from the bin until a second ball of the same colour as the first one drawn appears, what is probability that a total of at most five balls will be drawn?

SOLUTIONS. **a.** If Z is a continuous random variable with an exponential distribution with parameter λ, then, by definition,

$$P(Z \leq 2) = \int_{infty}^{2} h(z)\,dz = \int_{-\infty}^{0} 0\,dz + \int_{0}^{2} \lambda e^{-\lambda z}\,dz \qquad \text{Let } u = -\lambda z, \text{ so } du = -\lambda dz \text{ and } \lambda\,dz = (-1)\,du.$$

$$= 0 + \int_{z=0}^{z=2} e^{u}(-1)\,du = (-1)e^{u}\big|_{z=0}^{z=2} = (-1)e^{-\lambda z}\big|_{0}^{2}$$

$$= (-1)e^{-\lambda\cdot 2} - (-1)e^{-\lambda\cdot 0} = -e^{-2\lambda} + e^{0} = 1 - e^{-2\lambda}\,.$$

It follows that we need to solve the equation $1 - e^{-2\lambda} = P(Z \leq 2) = \frac{1}{2}$ for λ:

$$1 - e^{-2\lambda} = \frac{1}{2} \implies e^{-2\lambda} = \frac{1}{2} \implies -2\lambda = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) = (-1)\ln(2)$$

$$\implies \lambda = \frac{(-1)\ln(2)}{-2} = \frac{1}{2}\ln(2) = \ln\left(2^{1/2}\right) = \ln\left(\sqrt{2}\right) \approx 0.3466 \qquad \square$$

b. Consider how more than five balls might be drawn if one draws without replacement until a second ball of the same colour as the first drawn turns up. If the first ball drawn was blue, we might get the sequence $BRRRRB$ of length 6 or the sequence $BRRRRRB$ of length 7; we cannot get a longer sequence because there are only 5 red balls in the bin. Similarly, if the first ball drawn was red, $RBBBBR$ and $RBBBBBR$ are the only possibilities. The probability of getting a given colour of ball on a given draw is just the proportion of the number of balls of that colour among the balls remaining in the bin. It follows that

$$\begin{aligned} P(> 5 \text{ balls}) &= P(BRRRRB) + P(BRRRRRB) + P(RBBBBR) + P(RBBBBBR) \\ &= \frac{5}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{4}{5} + \frac{5}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}\cdot\frac{4}{4} \\ &\quad + \frac{5}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{4}{5} + \frac{5}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}\cdot\frac{4}{4} \\ &= \frac{1}{63} + \frac{1}{252} + \frac{1}{63} + \frac{1}{252} = \frac{10}{252} = \frac{5}{126} \approx 0.0397 \qquad \blacksquare \end{aligned}$$

[Total = 30]