Mathematics 1550H – Introduction to Probability
TRENT UNIVERSITY, Summer 2015

Quiz Solutions

Quiz #1. Wednesday, 24 June, 2015. *[10 minutes]*

A fair (standard six-sided) die and a fair (standard two-sided) coin are tossed simultaneously. Let b and c be the outcomes of the die and coin tosses respectively. Since the die and coin are fair, any pair (b,c) is as likely to turn up as any other.

1. What is the sample space Ω? How many possible outcomes are there in Ω? *[2]*
2. Let A be the event that b is odd and $c = T$. Find $P(A)$. *[3]*

SOLUTIONS. 1. There are $6 \cdot 2 = 12$ outcomes in Ω:

$$\begin{aligned}\Omega = \{\, &(1,H),\ (1,T),\ (2,H),\ (2,T),\ (3,H),\ (3,T),\\ &(4,H),\ (4,T),\ (5,H),\ (5,T),\ (6,H),\ (6,T)\,\} \qquad \square\end{aligned}$$

2. First, note that because $|\Omega| = 12$ and every outcome is as likely as any other, the probability of outcome (b,c) is $m(b,c) = \frac{1}{12}$ for every $(b,c) \in \Omega$. Now, if A is the event that b is odd and $c = T$, then $A = \{\, (1,T),\ (3,T),\ (5,T)\,\}$, so

$$P(A) = m(1,T) + m(3,T) + m(5,T) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4} = 0.25\,. \qquad \blacksquare$$

Quiz #2. Monday, 29 June, 2015. *[10 minutes]*

Do *one* of questions 1 and 2.

1. A fair coin is tossed until it comes up heads. What is the probability that it will be tossed at least four times? *[5]*
2. Suppose X is a continuous random variable with a uniform distribution on $[0,2]$, so it has $f(x) = \begin{cases} \frac{1}{2} & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$ as its probability density function. Compute $P\left(X \le \frac{3}{2}\right)$. *[5]*

SOLUTIONS. 1. Here is the brute force approach, which has the virtue of simplicity:

$$\begin{aligned}P(\ge 4 \text{ tosses until } H) &= 1 - P(\le 3 \text{ tosses until } H) = 1 - [P(H) + P(TH) + P(TTH)]\\ &= 1 - \left[\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right] = 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right] = 1 - \frac{7}{8} = \frac{1}{8} \qquad \square\end{aligned}$$

2. By definition, since we are dealing with a continuous distribution with density function $f(x)$:

$$\begin{aligned}P\left(X \le \frac{3}{2}\right) &= \int_{-\infty}^{3/2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{3/2} \frac{1}{2}\,dx = 0 + \frac{1}{2}x\Big|_0^{3/2}\\ &= \frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot 0 = \frac{3}{4} - 0 = \frac{3}{4} \qquad \blacksquare\end{aligned}$$

Quiz #3. Monday, 6 July, 2015. *[12 minutes]*

1. You are given ten books, eight of which are unique and the other two are indistinguishable from each other, though the two identical ones are different from the other eight. How many ways are there to arrange the ten books on two shelves, each of which could hold all ten books? *[5]*

SOLUTION. Suppose instead that the ten books were all distinguishable from each other. Then the number of ways to arrange them would be equal to the number of ways of putting the ten books in order with a single divider: the books before the divider go on the first shelf and those after it go on the second shelf, with the order of the books being maintaned otherwise. There are $(10+1)! = 11!$ ways to arrange ten distinct books and one divider, so there are $11!$ ways to arrange ten distinct books on two shelves.

Each arrangement of ten books, eight distinct and two identical, corresponds to two arrangements in which the ten books are all distinct, because swapping the two identical books leaves you with the same arrangement, but would give you two different if the two identical books were distinct instead. It follows that there are $11!/2 = 11 \cdot 10 \cdot \cdots \cdot 4 \cdot 3 = 19,958,400$ ways of arranging eight distinct and two identical books on two shelves. ■

Quiz #4. Wednesday, 8 July, 2015. *[15 minutes]*

A jar initially contains nine chocolate chip and seven butter cookies. While doing your probability homework, you eat five cookies, reaching into the jar and picking one at random each time. You are concentrating hard on your homework, so you don't really notice what kind you eat.

1. If the the last two of the five you eat are butter cookies, what is the probability that first three were chocolate chip cookies? *[5]*

SOLUTION. Let A be the event that the first three cookies eaten were all chocolate chip cookies and let B be the event that the the last two cookies eaten were butter cookies. $A \cap B$ is then the event that the first three cookies were chocolate chip cookies and the last two were butter cookies. To compute the probabilities of these events, note that we start with 9 chocolate chip and 7 butter cookies, for a total of sixteen. We thus have a probability of $\frac{9}{16}$ of picking a chocolate chip cookie and a probability of $\frac{7}{16}$ of picking a butter cookie for our first cookie. If we chose a chocolate chip cookie first, that would leave 8 chocolate chip and 7 butter cookies in the jar, for a total of 15, so we would have a probability of $\frac{8}{15}$ of picking a chocolate chip cookie and a probability of $\frac{7}{15}$ of picking a butter cookie for our first cookie, and so on. It follows that:

$$\begin{aligned}
P(A) &= \frac{9}{16} \cdot \frac{8}{15} \cdot \frac{7}{14} = \frac{3}{20} = 0.15 \\
P(A \cap B) &= \frac{9}{16} \cdot \frac{8}{15} \cdot \frac{7}{14} \cdot \frac{7}{13} \cdot \frac{6}{12} = \frac{21}{520} \approx 0.0404 \\
P(B) &= \frac{7}{16} \cdot \frac{6}{15} = \frac{7}{40} = 0.175
\end{aligned}$$

[The interested reader may care to contemplate why, if B is the event that the fourth and fifth cookies to be eaten are butter cookies, that the calculation of $P(B)$ is done as if they were the first and second. It isn't wrong. Why not?]

Thus $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{21}{520}}{\frac{7}{40}} = \dfrac{21}{520} \cdot \dfrac{40}{7} = \dfrac{3}{13} \approx 0.2308$. As an aside, note that since $P(A|B) \neq P(A)$, A and B are not independent. ■

Quiz #5. Wednesday, 15 July, 2015. *[12 minutes]*

A fair die is rolled five times. Let X be the number of times that the roll came up 3 or 6.

1. What is the probability function of X? *[2.5]*
2. Compute the expected value $E(X)$. *[2.5]*

SOLUTIONS. 1. Let's call having a roll come up 3 or 5 a success and call it a failure otherwise. Then each roll is a Bernoulli trial with the probability of success being $p = \frac{2}{6} = \frac{1}{3}$ and of failure being $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$. Counting the number of successes in five trials gives us a binomial distribution with $n = 5$ and p and q as above. Thus the probability function of X is $m(k) = \binom{5}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{5-k}$, where $0 \leq k \leq 5$. □

2. As noted in class and text, the expected value of a random variable with a binomial distribution with parameters n and p is np. Thus $E(X) = 5 \cdot \dfrac{1}{3} = \dfrac{5}{3}$. ■

Quiz #6. Monday, 20 July, 2015. *[15 minutes]*

1. Suppose the continuous random variable X has probability density $f(x) = \begin{cases} 3x^{-4} & x \geq 1 \\ 0 & x < 1 \end{cases}$.

   Compute the expected value $E(X)$ and variance $V(X)$ of X. *[5]*

SOLUTION. We plug the probability density function we are given into the definitions of $E(X)$ and $V(X)$ for continuous random variables and calculate away:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{1} x \cdot 0\,dx + \int_{1}^{\infty} x \cdot 3x^{-4}\,dx = 0 + \int_{1}^{\infty} 3x^{-3}\,dx \\
&= 3\frac{x^{-2}}{-2}\bigg|_1^\infty = -\frac{3}{2x^2}\bigg|_1^\infty = \left(-\frac{3}{2\infty^2}\right) - \left(-\frac{3}{2 \cdot 1^2}\right) = -0 + \frac{3}{2} = \frac{3}{2} \\
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{1} x^2 \cdot 0\,dx + \int_{1}^{\infty} x^2 \cdot 3x^{-4}\,dx = 0 + \int_{1}^{\infty} 3x^{-2}\,dx \\
&= 3\frac{x^{-1}}{-1}\bigg|_1^\infty = -\frac{3}{x}\bigg|_1^\infty = \left(-\frac{3}{\infty}\right) - \left(-\frac{3}{1}\right) = -0 + 3 = 3 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 3 - \left[\frac{3}{2}\right]^2 = 3 - \frac{9}{4} = \frac{3}{4} \qquad ■
\end{aligned}$$

Quiz #8. Wednesday, 22 July, 2015. *[15 minutes]*

1. Suppose X and Y are independent discrete random variables which have the same probability function, namely $m(k) = \begin{cases} \frac{1}{8} & k=0 \\ \frac{1}{2} & k=1 \\ \frac{3}{8} & k=2 \\ 0 & \text{otherwise} \end{cases}$. Find the probability function of $Z = X + Y$. *[5]*

SOLUTION. The probability function of Z is the convolution of the probability functions of X and Y, that is, the convolution of $m(k)$ with itself. If j is an integer, then:

$$\begin{aligned}
(m * m)(j) &= \sum_{k=-\infty}^{\infty} m(k)m(j-k) \\
&= \cdots + m(-2)m\left(j-(-2)\right) + m(-1)m\left(j-(-1)\right) + m(0)m\left(j-0\right) + m(1)m\left(j-1\right) \\
&\qquad + m(2)m\left(j-2\right) + m(3)m\left(j-3\right) + m(4)m\left(j-4\right) + \cdots \\
&= \cdots + 0m\left(j-(-2)\right) + 0m\left(j-(-1)\right) + m(0)m\left(j-0\right) + m(1)m\left(j-1\right) \\
&\qquad + m(2)m\left(j-2\right) + 0m\left(j-3\right) + 0m\left(j-4\right) + \cdots \\
&= m(0)m\left(j-0\right) + m(1)m\left(j-1\right) + m(2)m\left(j-2\right) \\
&= \frac{1}{8}m\left(j\right) + \frac{1}{2}m\left(j-1\right) + \frac{3}{8}m\left(j-2\right)
\end{aligned}$$

Note that it follows that $(m * m)(j) \neq 0$ only when at least one of j, $j-1$, or $j-2$ is equal to 0, 1, or 2; that is, when $j = 0$, 1, 2, 3, or 4. Thus

$$\begin{aligned}
(m * m)(0) &= \frac{1}{8}m\left(0\right) + \frac{1}{2}m\left(0-1\right) + \frac{3}{8}m\left(0-2\right) = \frac{1}{8}\cdot\frac{1}{8} + \frac{1}{2}\cdot 0 + \frac{3}{8}\cdot 0 = \frac{1}{64} \\
(m * m)(1) &= \frac{1}{8}m\left(1\right) + \frac{1}{2}m\left(1-1\right) + \frac{3}{8}m\left(1-2\right) = \frac{1}{8}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{8} + \frac{3}{8}\cdot 0 = \frac{1}{8} \\
(m * m)(2) &= \frac{1}{8}m\left(2\right) + \frac{1}{2}m\left(2-1\right) + \frac{3}{8}m\left(2-2\right) = \frac{1}{8}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{1}{2} + \frac{3}{8}\cdot\frac{1}{8} = \frac{5}{16} \\
(m * m)(3) &= \frac{1}{8}m\left(3\right) + \frac{1}{2}m\left(3-1\right) + \frac{3}{8}m\left(3-2\right) = \frac{1}{8}\cdot 0 + \frac{1}{2}\cdot\frac{3}{8} + \frac{3}{8}\cdot\frac{1}{2} = \frac{3}{8} \\
(m * m)(4) &= \frac{1}{8}m\left(4\right) + \frac{1}{2}m\left(4-1\right) + \frac{3}{8}m\left(4-2\right) = \frac{1}{8}\cdot 0 + \frac{1}{2}\cdot 0 + \frac{3}{8}\cdot\frac{3}{8} = \frac{9}{64}
\end{aligned}$$

(and $(m * m)(j) = 0$ if $j \neq 0$, 1, 2, 3, or 4) is the probability function of $Z = X + Y$. ■

Quiz #9. Monday, 27 July, 2015. *[15 minutes]*

1. Suppose X is a random variable with expected value $5 = E(X)$ and variance $4 = \sigma^2 = V(X)$, which is symmetric about its mean, that is, $P(X - 5 \leq -a) = P(X - 5 \geq a)$ for all $a > 0$. Use the symmetry of X and Chebyshev's Inequality to estimate $P(X \geq 13)$. *[5]*

SOLUTION. Recall that Chebyshev's Inequality states that $P\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2}$, where $\mu = E(X)$, $\sigma = \sqrt{V(X)}$, and $k > 0$ is a real number. Note that $|X - \mu| \geq k\sigma$ exactly when either $X - \mu \geq k\sigma$ or $-(X - \mu) \geq k\sigma$, *i.e.* $X - \mu \geq k\sigma$ or $X - \mu \leq -k\sigma$, and we obviously can't have both of these be true at once. It follows that

$$P\left(|X - \mu| \geq k\sigma\right) = P\left(X - \mu \geq k\sigma\right) + P\left(X - \mu \leq -k\sigma\right).$$

In our case, $\mu = E(X) = 5$ and $\sigma = \sqrt{V(X)} = \sqrt{4} = 2$, and $X \geq 13$ exactly $X - \mu = X - 5 \geq 8 = 4 \cdot 2 = 4\sigma$. Thus

$$\begin{aligned}
P(X \geq 13) = P(X - 5 \geq 4 \cdot 2) &= \frac{1}{2}\left[2P(X - 5 \geq 4 \cdot 2)\right] \\
&= \frac{1}{2}\left[P(X - 5 \geq 4 \cdot 2) + P(X - 5 \geq 4 \cdot 2)\right] \\
&= \frac{1}{2}\left[P(X - 5 \geq 4 \cdot 2) + P(-4 \cdot 2 \leq X - 5)\right] \quad \text{(by the symmetry of } X\text{)} \\
&= \frac{1}{2}P\left(|X - 5| \geq 4 \cdot 2\right) \\
&\leq \frac{1}{2} \cdot \frac{1}{4^2} = \frac{1}{2} \cdot \frac{1}{16} = \frac{1}{32},
\end{aligned}$$

i.e. $P(X \geq 13) \leq \frac{1}{32} = 0.03125$. Note that this estimate is actually an upper bound for the probability. ■

Quiz #10. Wednesday, 29 July, 2015. *[15 minutes]*

Suppose the discrete random variables X and Y are jointly distributed, with their joint probability function $p(x, y)$ being given by the table below.

$X \backslash Y$	1	2
0	0.2	0.1
2	0.2	0.2
4	0.1	0.2

1. Compute $\text{Cov}(X, Y)$, the covariance of X and Y. *[5]*

SOLUTION. By definition, $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$, so we first compute the expected values in question. Let p_X and p_Y denote the probability functions for X and Y, taken individually; then $p_X(x) = \sum_y p(x, y)$ and $p_Y(y) = \sum_x p(x, y)$. Then

$$\begin{aligned}
E(X) = \sum_x x p_X(x) = \sum_x x \left[\sum_y p(x, y)\right] &= 0\,[0.2 + 0.2] + 2\,[0.2 + 0.2] + 4\,[0.1 + 0.2] \\
&= 0 + 0.8 + 1.2 = 2\,, \\
E(Y) = \sum_y y p_Y(y) = \sum_y y \left[\sum_x p(x, y)\right] &= 1\,[0.2 + 0.2 + 0.1] + 2\,[0.1 + 0.2 + 0.2] \\
&= 0.5 + 1 = 1.5\,,
\end{aligned}$$

and

$$\begin{aligned}
E(XY) &= \sum_x \sum_y xyp(x, y) \\
&= 0 \cdot 1 \cdot 0.2 + 0 \cdot 2 \cdot 0.1 + 2 \cdot 1 \cdot 0.2 + 2 \cdot 2 \cdot 0.2 + 4 \cdot 1 \cdot 0.1 + 4 \cdot 2 \cdot 0.2 \\
&= 0 + 0 + 0.4 + 0.8 + 0.4 + 1.6 = 3.2\,,
\end{aligned}$$

so $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 3.2 - 2 \cdot 1.5 = 3.2 - 3 = 0.2$. ■