Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2015

FINAL EXAMINATION
Tuesday, 4 August, 2015

Space-Time: 14:00–17:00 in GCS 103 *Courtesy of* Стефан Біланюк.

Instructions: Do both of parts **E** and **V**, and, if you wish, part **Cov**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; ≤ 1 brain.

Part E. Do all of **1**–**5**. *[Subtotal = 70/100]*

1. A fair coin is tossed and then repeatedly tossed until it comes up with a different face from the one that came up on the first toss.

a. What are the sample space and probability function? *[7]*

b. Let A be the event that four tosses took place and let B be the event that the first toss was a head. Compute $P(A|B)$. *[8]*

2. A fair die is rolled twice. Let A be the event that the sum of the two rolls is more than seven, and let B be the event that one of the rolls was odd and the other even.

a. What are the sample space and probability function? *[5]*

b. Compute $P(A)$ and $P(B)$. *[5]*

c. Determine whether A and B are independent or not. *[5]*

3. Suppose X is a normally distributed continuous random variable with expected value $\mu = 10$ and standard deviation $\sigma = 5$.

a. Use Chebyshev's Inequality to estimate $P\left(|X - 10| \geq 10\right)$. *[5]*

b. Compute $P\left(|X - 10| \geq 10\right)$ using a standard normal table. *[5]*

4. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck.

a. What is the probability that the hand is a *flush,* that is, that all five cards are from the same suit? *[5]*

b. What is the probability that exactly three of the four suits are represented in the hand? *[10]*

5. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{3}{2}x^2 & -1 \leq x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases} .$$

a. Verify that $f(x)$ is indeed a probability density. *[6]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[9]*

*[Parts **V** and **Cov** are on page 2.]*

Part V. Do any *two* (2) of **6–9**. *[Subtotal = 30/100]*

6. Let $g(x) = \begin{cases} 2xe^{-x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$ be the probability density function of the continuous random variable X.

- **a.** Verify that $g(x)$ is indeed a probability density function. *[7]*
- **b.** Find the *median* of X, *i.e.* the number m such that $P(X \leq m) = \frac{1}{2} = 0.5$. *[8]*

7. Suppose that the independent discrete random variables X and Y are identically and uniformly distributed, each with probability function $m(k) = \begin{cases} \frac{1}{4} & k = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$, and let $Z = X + Y$. Find the probability function of Z. *[15]*

8. A jar contains 8 white beads and 16 black beads. Beads are chosen randomly from the jar until the third time a white bead turns up.

- **a.** How many beads would you expect to have to choose if each bead is replaced before the next is chosen? *[8]*
- **b.** If beads are not replaced before the next is chosen, what is the maximum number of beads that could be chosen before the third white bead turns up? What is the probability of this event? *[7]*

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	-1	0	1
1	0.1	0	0.2
2	0.2	0.1	0.1
3	0.1	0.1	0.1

- **a.** Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\text{Cov}(X, Y)$ of X and Y. *[10]*
- **b.** Let $W = X - 2Y$. Compute $E(W)$ and $V(W)$. *[5]*

[Total = 100]

Part Cov. Bonus!

σ. One hundred people line up to board an airplane. Each has a boarding pass with an assigned seat. However, the first person to board has lost their boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his or her assigned seat? *[1]*

σ^2. Write an original little poem about probability or mathematics in general. *[1]*

I HOPE THAT YOU ENJOYED THE COURSE
AS MUCH AS YOU DO THE REST OF THE SUMMER!