Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2015

FINAL EXAMINATION
Tuesday, 4 August, 2015

Space-Time: 14:00–17:00 in GCS 103 *Courtesy of* Стефан Біланюк.

Instructions: Do both of parts **E** and **V**, and, if you wish, part **Cov**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; ≤ 1 brain.

Part E. Do all of **1**–**5**. *[Subtotal = 70/100]*

1. A fair coin is tossed and then repeatedly tossed until it comes up with a different face from the one that came up on the first toss.

- **a.** What are the sample space and probability function? *[7]*
- **b.** Let A be the event that four tosses took place and let B be the event that the first toss was a head. Compute $P(A|B)$. *[8]*

SOLUTIONS. **a.** The sample space is:

$$\Omega = \{ HT, TH, HHT, TTH, HHHT, TTTH, HHHHT, TTTTH, \dots \}$$

Since the coin is fair, $P(H) = P(T) = \frac{1}{2}$ for each toss of the coin, so the probability function is given by

$$m\left(H^kT\right) = m\left(T^kH\right) = \left(\frac{1}{2}\right)^k \frac{1}{2} = \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2^{k+1}},$$

where $k \geq 1$ is the number of times the first face actually comes up before the other one does. □

b. $B = \{ HT, HHT, HHHT, \dots \}$ is the event that the first toss was a head, so $P(B) = P(H) = \dfrac{1}{2}$. $A = \{ HHHT, TTTH \}$, so $A \cap B = \{ HHHT \}$, and thus $P(A \cap B) = m(HHHT) = \dfrac{1}{2^{3+1}} = \dfrac{1}{16}$. It follows that $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{16}}{\frac{1}{2}} = \dfrac{1}{16} \cdot \dfrac{2}{1} = \dfrac{1}{8}$. ■

2. A fair die is rolled twice. Let A be the event that the sum of the two rolls is more than seven, and let B be the event that one of the rolls was odd and the other even.

- **a.** What are the sample space and probability function? *[5]*
- **b.** Compute $P(A)$ and $P(B)$. *[5]*
- **c.** Determine whether A and B are independent or not. *[5]*

SOLUTIONS. **a.** There are several possible choices for a sample space and probability function here, but the most basic would be taking the sample space to be all the possible outcomes (a, b) of the two rolls:

$$\begin{aligned} \Omega &= \{ (a, b) \mid a,\, b = 1,\, 2,\, 3,\, 4,\, 5,\, 6 \} \\ &= \{ (1, 1),\, (1, 2), \dots,\, (6, 5),\, (6, 6) \} \end{aligned}$$

There are $6 \cdot 6 = 36$ possible outcomes, since the die is fair, each is equally likely, so $m(a,b) = \frac{1}{36}$ for each outcome $(a,b) \in \Omega$. □

b. Consider the following table giving the sum of the two rolls for each outcome:

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Since A be the event that the sum of the two rolls is more than seven, it follows from the table that

$$A = \{ (2,6), (3,5), (3,6), (4,4), (4,5), (4,6), (5,3), (5,4), (5,5), (5,6), (6,2), (6,3), (6,4), (6,5), (6,6) \}.$$

As each outcome is equally likely, $P(A) = \dfrac{\#\text{ outcomes in } A}{\#\text{ outcomes in } \Omega} = \dfrac{15}{36} = \dfrac{5}{12}$.

One could also count outcomes to compute $P(B)$, but it is probably easier to observe that a single roll is equally likely to come up odd (1, 3, or 5) as even (2, 4, or 6), so $P(\text{odd}) = P(\text{even}) = \frac{1}{2}$. It follows that $P(B) = P(\text{odd})P(\text{even}) + P(\text{even})\P(\text{odd}) = \dfrac{1}{2} \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2}$. □

c. A and B are independent exactly when $P(A \cap B) = P(A)P(B)$. We know that $P(A) = \frac{5}{12}$ and $P(B) = \frac{1}{2}$ from the answer to **b** above. Inspecting the table and/or the listing of A used in the answer to **b**, $A \cap B = \{ (3,6), (4,5), (5,4), (5,6), (6,5) \}$, so $P(A \cap B) = \dfrac{\#\text{ outcomes in } A \cap B}{\#\text{ outcomes in } \Omega} = \dfrac{5}{36}$. As $P(A \cap B) = \dfrac{5}{36} \neq \dfrac{5}{24} = \dfrac{5}{12} \cdot \dfrac{1}{2} = P(A)P(B)$, it follows that A and B are dependent, *i.e.* not independent. ■

3. Suppose X is a normally distributed continuous random variable with expected value $\mu = 10$ and standard deviation $\sigma = 5$.

a. Use Chebyshev's Inequality to estimate $P(|X - 10| \geq 10)$. *[5]*

b. Compute $P(|X - 10| \geq 10)$ using a standard normal table. *[5]*

Solutions. **a.** Chebyshev's Inequality tells us that $P(|X - \mu| \geq k\sigma) \leq \dfrac{1}{k^2}$. We have $\mu = 10$ and $\sigma = 5$, so

$$P(|X - 10| \geq 10) = P(|X - 10| \geq 2 \cdot 5) = P(|X - \mu| \geq 2\sigma) \leq \frac{1}{2^2} = \frac{1}{4}. \quad \square$$

b. Let $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-10}{5}$. Then the random variable Z has a standard normal distribution and $|X-10| \geq 10 \Leftrightarrow \left|\dfrac{X-10}{5}\right| \geq 2 \Leftrightarrow |Z| \geq 2$. Thus

$$\begin{aligned} P(|X-10| \geq 10) &= P(|Z| \geq 2) = P(Z \leq -2) + P(Z \geq 2) \\ &= P(Z \leq -2) + [1 - P(Z < 2)] \\ &\approx 0.0228 + [1 - 0.9772] = 0.0228 + 0.0228 = 0.0456, \end{aligned}$$

using the cumulative normal table to get values for $P(Z \leq -2)$ and $P(Z < 2)$. ■

4. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck.

a. What is the probability that the hand is a *flush,* that is, that all five cards are from the same suit? *[5]*

b. What is the probability that exactly three of the four suits are represented in the hand? *[10]*

Solutions. **a.** There are $\binom{52}{5}$ (unordered) hands of five cards that can be chosen from a standard 52-card deck. Each suit has thirteen cards, so there are $\binom{13}{5}$ (unordered) hands of five cards that can be chosen from each suit. Since there are four suits and each five-card hand is as likely to be chosen as any other,

$$P(\text{flush}) = \frac{(\#\text{ suits})(\#\text{ hands from each suit})}{\#\text{ hands}} = \frac{4\binom{13}{5}}{\binom{52}{5}} = \frac{99}{49980} \approx 0.00198\,. \quad \square$$

b. As noted above in the solution to part **a**, there are $\binom{52}{5}$ (unordered) hands of five cards that can be chosen from a standard 52-card deck. The problem here is how to count the number of hands which include cards from exactly three of the four suits. There are several ways to do this; the method used below was chosen because it is relatively straightforward to understand and do the computations for.

Note that a five-card hand with exactly three of the four suits represented in it must either have three cards from one suit and one card each from second and third suits, or have two cards from one suit, two more cards from a second suit, and a single card from a third suit. We will count these possibilities separately:

First, the number of hands with three cards from one suit and one card each from second and third suits is $\binom{4}{1}\binom{13}{3}\binom{3}{2}\binom{13}{1}\binom{13}{1} = 4 \cdot \frac{13 \cdot 12 \cdot 11}{6} \cdot 3 \cdot 13^2 = 580008$: there are $\binom{4}{1}$ ways to pick the suit with three cards in the hand, $\binom{13}{3} = \frac{13!}{10!3!} = \frac{13 \cdot 12 \cdot 11}{6}$ ways to pick the three cards from the suit in the hand, $\binom{3}{2} = 3$ ways to pick the other two suits represented in the hand out of the remaining three suits, and $\binom{13}{1}\binom{13}{1} = 13^2$ ways to pick one card out of each of these suits.

Second, the number of hands with two cards from one suit, two more cards from a second suit, and a single card from a third suit is $\binom{4}{2}\binom{13}{2}\binom{13}{2}\binom{2}{1}\binom{13}{1} = 6 \cdot \left(\frac{13 \cdot 12}{2}\right)^2 \cdot 2 \cdot 13 =$

949104: there are $\binom{4}{2} = 6$ ways to pick the two suits that have two cards each in the hand, $\binom{13}{2}\binom{13}{2} = \left(\frac{13!}{11!2!}\right)^2 = \left(\frac{13\cdot 12}{2}\right)^2$ ways to pick the two cards from each of the two suits in the hand, $\binom{2}{1} = 2$ to pick the other suit represented in the hand from the remaining two suits, and $\binom{13}{1} = 13$ ways to pick one card from this suit.

Since each hand is as likely to be chosen as any other and there are $\binom{52}{5} = 2598960$ ways to choose a five-card hand from a standard deck, it follows that

$$\begin{aligned} P(\text{exactly 3 suits in the hand}) &= \frac{\binom{4}{1}\binom{13}{3}\binom{3}{2}\binom{13}{1}\binom{13}{1} + \binom{4}{2}\binom{13}{2}\binom{13}{2}\binom{2}{1}\binom{13}{1}}{\binom{52}{5}} \\ &= \frac{580008 + 949104}{2598960} = \frac{1529112}{2598960} \approx 0.58836\,. \quad \blacksquare \end{aligned}$$

5. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{3}{2}x^2 & -1 \le x \le 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}.$$

a. Verify that $f(x)$ is indeed a probability density. *[6]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[9]*

Solutions. **a.** First, since $\frac{3}{2}x^2 \ge 0$ for all $x \in [-1,1]$ and $0 \ge 0$ for all $x \notin [-1,1]$, we have that $f(x) \ge 0$ for all $x \in \mathbb{R}$, as required. Second,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{2}x^2\,dx + \int_{1}^{\infty} 0\,dx = 0 + \frac{3}{2}\cdot\left.\frac{x^3}{3}\right|_{-1}^{1} + 0 \\ &= \left.\frac{x^3}{2}\right|_{-1}^{1} = \frac{1^3}{2} - \frac{(-1)^3}{2} = \frac{1}{2} - \frac{-1}{2} = \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Thus $f(x)$ satisfies both requirements for a function $\mathbb{R} \to \mathbb{R}$ to be a probability density. $\square$

b. We plug into the definitions and integrate away.

$$\begin{aligned} \text{First,} \quad E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{-1} x\cdot 0\,dx + \int_{-1}^{1} x\cdot\frac{3}{2}x^2\,dx + \int_{1}^{\infty} x\cdot 0\,dx \\ &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{2}x^3\,dx + \int_{1}^{\infty} 0\,dx = 0 + \frac{3}{2}\cdot\left.\frac{x^4}{4}\right|_{-1}^{1} + 0 \\ &= \frac{3\cdot 1^4}{8} - \frac{3\cdot(-1)^4}{8} = \frac{3}{8} - \frac{3}{8} = 0. \end{aligned}$$

$$\begin{aligned} \text{Second,} \quad V(X) &= E\left(X^2\right) - [E(X)]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - 0^2 \\ &= \int_{-\infty}^{-1} x^2\cdot 0\,dx + \int_{-1}^{1} x^2\cdot\frac{3}{2}x^2\,dx + \int_{1}^{\infty} x^2\cdot 0\,dx - 0 \\ &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{2}x^4\,dx + \int_{1}^{\infty} 0\,dx = 0 + \frac{3}{2}\cdot\left.\frac{x^5}{5}\right|_{-1}^{1} + 0 \\ &= \frac{3\cdot 1^5}{10} - \frac{3\cdot(-1)^5}{10} = \frac{3}{10} - \frac{-3}{10} = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}. \quad \blacksquare \end{aligned}$$

Part V. Do any *two* (2) of **6–9**. *[Subtotal = 30/100]*

6. Let $g(x) = \begin{cases} 2xe^{-x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(x)$ is indeed a probability density function. *[7]*

b. Find the *median* of X, *i.e.* the number m such that $P(X \leq m) = \frac{1}{2} = 0.5$. *[8]*

SOLUTIONS. **a.** First, when $x \geq 0$, $2xe^{-x^2} \geq 0$, and when $x < 0$, $0 \geq 0$, so $g(x) \geq 0$ for all $x \in \mathbb{R}$. Second, using the substitution $u = -x^2$, so $du = -2x\,dx$ and thus $2x\,dx = (-1)\,du$ and $\begin{matrix} x & 0 & \infty \\ u & 0 & -\infty \end{matrix}$, we have:

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx = 0 + \int_{0}^{-\infty} e^u(-1)\,du$$
$$= \int_{-\infty}^{0} e^u\,du = e^u\big|_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1$$

Thus $g(x)$ satisfies both requirements for a function $\mathbb{R} \to \mathbb{R}$ to be a probability density. $\square$

b. By defintion, $P(X \leq m) = \int_{-\infty}^{m} g(x)\,dx$. Since $g(x) = 0$ when $x < 0$, if $m < 0$, we would have $P(X \leq m) = \int_{-\infty}^{m} 0\,dx = 0$. As we are supposed to find an m so that $P(X \leq m) = \frac{1}{2}$, such an m must be ≥ 0. For an $m \geq 0$, we have, using the substitution $u = -x^2$ just as in the solution to **a** above, except that the limits work out to be $\begin{matrix} x & 0 & m \\ u & 0 & -m^2 \end{matrix}$:

$$P(X \leq m) = \int_{-\infty}^{m} g(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{m} 2xe^{-x^2}\,dx = 0 + \int_{0}^{-m^2} e^u(-1)\,du$$
$$= \int_{-m^2}^{0} e^u\,du = e^u\big|_{-m^2}^{0} = e^0 - e^{-m^2} = 1 - e^{-m^2}$$

Since we want an m such that $P(X \leq m) = \frac{1}{2}$, we need to solve the equation $1 - e^{-m^2} = \frac{1}{2}$ for m:

$$1 - e^{-m^2} = \frac{1}{2} \implies e^{-m^2} = 1 - \frac{1}{2} = \frac{1}{2} \implies -m^2 = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) = -\ln(2)$$
$$\implies m^2 = \ln(2) \implies m = \sqrt{\ln(2)}$$

Note that we need the positive root of ln(2) here because we have already know from our work above that $m \geq 0$. Thus the median of X is $m = \sqrt{\ln(2)}$. ■

7. Suppose that the independent discrete random variables X and Y are identically and uniformly distributed, each with probability function $m(k) = \begin{cases} \frac{1}{4} & k = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$, and let $Z = X + Y$. Find the probability function of Z. *[15]*

Solution. X and Y are independent and identically distributed, with probability function $m(k)$, so the probability function of $X + Y$ is the convolution of the probability function $m(k)$ with itself, namely $(m * m)(j) = \sum_{k=-\infty}^{\infty} m(k)m(j-k)$. Hence, if j is an integer,

$$\begin{aligned}
(m * m)(j) &= \cdots + m(0)m(j-0) + m(1)m(j-1) + m(2)m(j-2) \\
&\qquad + m(3)m(j-3) + m(4)m(j-4) + m(5)m(j-5) + \cdots \\
&= \cdots + 0m(j-0) + m(1)m(j-1) + m(2)m(j-2) \\
&\qquad + m(3)m(j-3) + m(4)m(j-4) + 0m(j-5) \cdots \\
&= m(1)m(j-1) + m(2)m(j-2) + m(3)m(j-3) + m(4)m(j-4) \\
&= \frac{1}{4}m(j-1) + \frac{1}{4}m(j-2) + \frac{1}{4}m(j-3) + \frac{1}{4}m(j-4).
\end{aligned}$$

Note that it follows that $(m * m)(j) \neq 0$ only when at least one of $j-1$, $j-2$, $j-3$, or $j-4$ is equal to 1, 2, 3, or 4; that is, when $j = 2$, 3, 4, 5, 6, 7, or 8. Thus

$$\begin{aligned}
(m * m)(2) &= \frac{1}{4}m(2-1) + \frac{1}{4}m(2-2) + \frac{1}{4}m(2-3) + \frac{1}{4}m(2-4) \\
&= \frac{1}{4}m(1) + \frac{1}{4}m(0) + \frac{1}{4}m(-1) + \frac{1}{4}m(-2) = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 = \frac{1}{16} \\
(m * m)(3) &= \frac{1}{4}m(3-1) + \frac{1}{4}m(3-2) + \frac{1}{4}m(3-3) + \frac{1}{4}m(3-4) \\
&= \frac{1}{4}m(2) + \frac{1}{4}m(1) + \frac{1}{4}m(0) + \frac{1}{4}m(-1) = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 = \frac{1}{8} \\
(m * m)(4) &= \frac{1}{4}m(4-1) + \frac{1}{4}m(4-2) + \frac{1}{4}m(4-3) + \frac{1}{4}m(4-4) \\
&= \frac{1}{4}m(3) + \frac{1}{4}m(2) + \frac{1}{4}m(1) + \frac{1}{4}m(0) = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 = \frac{3}{16} \\
(m * m)(5) &= \frac{1}{4}m(5-1) + \frac{1}{4}m(5-2) + \frac{1}{4}m(5-3) + \frac{1}{4}m(5-4) \\
&= \frac{1}{4}m(4) + \frac{1}{4}m(3) + \frac{1}{4}m(2) + \frac{1}{4}m(1) = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4} \\
(m * m)(6) &= \frac{1}{4}m(6-1) + \frac{1}{4}m(6-2) + \frac{1}{4}m(6-3) + \frac{1}{4}m(6-4) \\
&= \frac{1}{4}m(5) + \frac{1}{4}m(4) + \frac{1}{4}m(3) + \frac{1}{4}m(2) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{16} \\
(m * m)(7) &= \frac{1}{4}m(7-1) + \frac{1}{4}m(7-2) + \frac{1}{4}m(7-3) + \frac{1}{4}m(7-4) \\
&= \frac{1}{4}m(6) + \frac{1}{4}m(5) + \frac{1}{4}m(2) + \frac{1}{4}m(3) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{8} \\
(m * m)(8) &= \frac{1}{4}m(8-1) + \frac{1}{4}m(8-2) + \frac{1}{4}m(8-3) + \frac{1}{4}m(8-4) \\
&= \frac{1}{4}m(7) + \frac{1}{4}m(6) + \frac{1}{4}m(5) + \frac{1}{4}m(4) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16},
\end{aligned}$$

and $(m * m)(j) = 0$ otherwise, is the probability function of $Z = X + Y$. ■

8. A jar contains 8 white beads and 16 black beads. Beads are chosen randomly from the jar until the third time a white bead turns up.

a. How many beads would you expect to have to choose if each bead is replaced before the next is chosen? *[8]*

b. If beads are not replaced before the next is chosen, what is the maximum number of beads that could be chosen before the third white bead turns up? What is the probability of this event? *[7]*

SOLUTIONS. **a.** There are a total of $24 = 8 + 16$ beads. If each bead is replaced before the next is chosen randomly, then the probabilities of picking beads of each colour are given by $P(\text{white}) = \frac{8}{24} = \frac{1}{3}$ and $P(\text{black}) = \frac{16}{24} = \frac{2}{3}$ every time, *i.e.* it is a Bernoulli trial with $p = \frac{1}{3}$ [counting a white bead as a success] and $q = 1 - p = \frac{2}{3}$. Picking repeatedly until the third time a white bead turns up therefore has a negative binomial distribution with $p = \frac{1}{3}$ and $k = 3$. A random variable with negative binomial distribution with these parameters has expected value $\mu = \frac{k}{p} = \frac{3}{1/3} = \frac{3 \cdot 3}{1} = 9$. □

b. If beads are not replaced before the next is chosen, the maximum number of beads that could be chosen before the third white bead turns up is $16 + 2 = 18$: all 16 black beads and 2 white ones would have to come up before the third white one does. The probability of this occurring is a little more difficult to compute because the probability of picking a bead of a given colour is the proportion of the beads of that colour remaining to the total number of beads remaining when the choice is made, and these numbers change if beads are not replaced after each selection.

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	-1	0	1
1	0.1	0	0.2
2	0.2	0.1	0.1
3	0.1	0.1	0.1

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\text{Cov}(X, Y)$ of X and Y. *[10]*

b. Let $W = X - 2Y$. Compute $E(W)$ and $V(W)$. *[5]*

SOLUTIONS. **a.** Off we go!

$$\begin{aligned}
E(X) &= 1(0.1+0+0.2)+2(0.2+0.1+0.1)+3(0.1+0.1+0.1)\\
&= 1\cdot 0.3+2\cdot 0.4+3\cdot 0.4 = 0.3+0.8+0.9 = 2\\
E(Y) &= (-1)(0.1+0.2+0.1)+0(0+0.1+0.1)+1(0.2+0.1+0.1)\\
&= -1\cdot 0.4+0\cdot 0.2+1\cdot 0.4 = -0.4+0+0.4 = 0\\
V(X) &= E\left(X^2\right)-[E(X)]^2\\
&= \left[1^2(0.1+0+0.2)+2^2(0.2+0.1+0.1)+3^2(0.1+0.1+0.1)\right]-2^2\\
&= [1\cdot 0.3+4\cdot 0.4+9\cdot 0.3]-4 = [0.3+1.6+2.7]-4 = 4.6-4 = 0.6\\
V(Y) &= E\left(Y^2\right)-[E(Y)]^2\\
&= \left[(-1)^2(0.1+0.2+0.1)+0^2(0+0.1+0.1)+1^2(0.2+0.1+0.1)\right]-0^2\\
&= [1\cdot 0.4+0\cdot 0.2+1\cdot 0.4]-0 = 0.4+0+0.4 = 0.8\\
\mathrm{Cov}(X,Y) &= E(XY)-E(X)E(Y) = \left[\sum_{x,y} xy\cdot m(x,y)\right]-0.6\cdot 0.8\\
&= [1\cdot(-1)\cdot 0.1+1\cdot 0\cdot 0+1\cdot 1\cdot 0.2+2\cdot(-1)\cdot 0.2+2\cdot 0\cdot 0.1+2\cdot 1\cdot 0.1\\
&\qquad +3\cdot(-1)\cdot 0.1+3\cdot 0\cdot 0.1+3\cdot 1\cdot 0.1]-0.48\\
&= [-0.1+0+0.2-0.4+0+0.2-0.3+0+0.3]-0.48\\
&= -0.1-0.48 = -0.58 \quad \square
\end{aligned}$$

b. Recall that if X and Y are random variables and a and b are constants, then $V(aX+bY) = a^2v(X)+b^2V(Y)+2ab\mathrm{Cov}(X,Y)$. Applying this formula, along with the information obtained in the solution to part **a** above, to $W = X-2Y$ yields:

$$\begin{aligned}
V(W) = V(X-2Y) &= 1^2V(X)+(-2)^2V(Y)+2\cdot 1(-2)\mathrm{Cov}(X,Y)\\
&= 1\cdot 0.6+4\cdot 0.8-4(-0.58) = 0.6+3.2+2.32 = 6.12 \quad \blacksquare
\end{aligned}$$

[Total = 100]

Part Cov. Bonus!

σ. One hundred people line up to board an airplane. Each has a boarding pass with an assigned seat. However, the first person to board has lost their boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his or her assigned seat? *[1]*

SOLUTION. Consider the situation when the kth passenger enters. None of the previous passengers showed any preference for the kth seat *vs.* the seat of the first passenger. This is, in particular, true when $k = n$. But the nth passenger can only occupy their own seat or the first passenger's seat. Therefore the probability is $\frac{1}{2} = 0.5$. [This problem is due to P. Winkler.] ■

σ^2. Write an original little poem about probability or mathematics in general. *[1]*

Solution. You're on your own on this one! ■

I hope that you enjoyed the course
as much as you do the rest of the summer!