TRENT UNIVERSITY, SUMMER 2014

MATH 1550H Test

14 July, 2014

Time: 50 minutes

Name: Solutions

STUDENT NUMBER: 2718281

Question	Mark
1	___
2	___
3	___
Bonus	___
Total	___ /30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

Bonus. If A, B, and C are events in a sample space S such that A and B are independent and B and C are independent, must A and C be independent too? Why or why not? *[1]*

SOLUTION. A and C need not be independent, though it is possible for them to be so.

For an example where A and C are not independent, suppose a fair coin is tossed twice. Let A be the event that the first toss is a H, B the event that the second toss is a H, and C be the event that the first toss is a T. It's pretty easy to check that A and B are independent and that B and C are independent, but A and C are complementary events and so cannot be independent.

For an example where A and C are independent, suppose a fair coin is tossed thrice. Let A be the event that the first toss is a H, B the event that the second toss is a H, and C be the event that the third toss is a H. It's pretty easy to check that very pair of these three events are independent. ■

1. Do any *three* (3) of **a**–**d**. *[12 = 3 × 4 each]*

a. Four cards are drawn at random, in order with replacement, from a standard 52-card deck. What is the probability that exactly two of the four cards are $\diamondsuit$s?

b. A fair standard die is rolled once. Let E be the event that an even number comes up, and F be the event that three or six comes up. Determine whether E and F are independent or not.

c. A fair coin is tossed repeatedly. What is the probability that at least two tails come up before the first head occurs?

d. A fair standard die is rolled repeatedly. What is the expected number of rolls before six comes up for the second time?

SOLUTIONS. **a.** When drawing a single card the probability of getting a $\diamondsuit$ is $\frac{13}{52} = \frac{1}{4}$; since we are drawing with replacement, this probability remains the same for each card drawn. It follows that if X counts the number of $\diamondsuit$s in four draws, X has a binomial distribution with $n = 4$ and $p = \frac{1}{4}$. Thus $P(2\ \diamondsuit\text{s}) = P(X = 2) = \binom{4}{2}\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{4-2} = 6 \cdot \frac{1}{16} \cdot \frac{9}{16} = \frac{27}{128} = 0.2109375$. $\square$

b. The sample space here is $S = \{1, 2, 3, 4, 5, 6\}$, all the outcomes of which are equally likely because the die is fair, and the events are $E = \{2, 4, 6\}$ and $F = \{3, 6\}$. $EF = E \cap F = \{6\}$, so

$$P(EF) = \frac{n(EF)}{n(S)} = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = \frac{3}{6} \cdot \frac{2}{6} = \frac{n(E)}{n(S)} \cdot \frac{n(F)}{n(S)} = P(E)P(F)\,,$$

from which it follows that E and F are indeed independent. $\square$

c. Let X be the number of tails that occur before the first head comes up. Since each toss is a Bernoulli trial with $p = \frac{1}{2}$, the coin being fair, the random variable X has a geometric distribution with $p = \frac{1}{2}$, so the probability of getting exactly x tails before the first head is $p(x) = P(X = x) = \left(1 - \frac{1}{2}\right)^x \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{x+1}$. It follows that the probability that at least two tails come up before the first head occurs is:

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left[\left(\frac{1}{2}\right)^{0+1} + \left(\frac{1}{2}\right)^{1+1}\right]$$

$$= 1 - \left[\frac{1}{2} + \left(\frac{1}{2}\right)^2\right] = 1 - \left[\frac{1}{2} + \frac{1}{4}\right] = 1 - \frac{3}{4} = \frac{1}{4} \qquad \square$$

d. The underlying experiment is essentially a Bernoulli trial where success is rolling a six, which has a probability $p = P(6) = \frac{1}{6}$ because the die is fair. If X counts the number of failures that occur before the second success, then X has a negative binomial distribution with parameters $r = 2$ and $p = \frac{1}{6}$. We know from class (*Don't we!?*) that a random variable with a negative binomial distribution with these parameters has expected value:

$$E(X) = \frac{r(1-p)}{p} = \frac{2\left(1 - \frac{1}{6}\right)}{\frac{1}{6}} = \frac{2 \cdot \frac{5}{6}}{\frac{1}{6}} = 10$$

However, we need the expected number of *rolls* before the second success, not just the number of failures, *i.e.* of $Y = X + 1$: $E(Y) = E(X + 1) = E(X) + 1 = 10 + 1 = 11$. $\blacksquare$

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. A fair coin is tossed repeatedly until either the second head or the second tail occurs. Let Y be the number of tosses required. Compute $E(Y)$ and $V(Y)$.

b. Show that if A and B are any events in a sample space S and $0 < P(B) < 1$, then $P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$.

c. All ten of one red, two blue, three green, and four yellow balls are arranged in a row. How many arrangements are there if balls of the same colour cannot be told apart?

Solutions. **a.** The second head or tail will occur on the second toss, if the first was a head or tail, respectively, and will otherwise occur on the third toss. That is, the sample space is $S = \{HH, TT, HTH, HTT, THH, THT\}$. Since the coin is fair, $P(H) = P(T) = \frac{1}{2}$ on any given toss, from which it follows that $P(HH) = P(TT) = \frac{1}{4}$ and $P(HTH) = P(HTT) = P(THH) = P(THT) = \frac{1}{8}$. It follows in turn that the probability function of Y is given by $p(2) = P(Y = 2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, $p(3) = P(Y = 3) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$, and $p(y) = P(Y = y) = 0$ if $y \neq 0$ or 1. Thus, by definition,

$$\begin{aligned} E(Y) &= 2p(2) + 3p(3) = 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = \frac{5}{2} = 2.5 \quad \text{and} \\ V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = \left(2^2 p(2) + 3^2 p(3)\right) - \left[\frac{5}{2}\right]^2 = \left(4 \cdot \frac{1}{2} + 9 \cdot \frac{1}{2}\right) - \frac{25}{4} \\ &= \frac{13}{2} - \frac{25}{4} = \frac{26}{4} - \frac{25}{4} = \frac{1}{4} = 0.25\,. \qquad \square \end{aligned}$$

b. First, observe that since AB and $A\bar{B}$ partition A, $P(A) = P(AB) + P(A\bar{B})$. Second, note that $0 < P(B) < 1$ implies that $P(\bar{B}) = 1 - P(B) > 1 - 1 = 0$. Since both $P(B)$ and $P(\bar{B})$ are > 0, we have that $P(A|B) = \dfrac{P(AB)}{P(B)}$ and $P(A|\bar{B}) = \dfrac{P(A\bar{B})}{P(\bar{B})}$. Hence $P(A) = P(AB) + P(A\bar{B}) = \frac{P(AB)}{P(B)} \cdot P(B) + \frac{P(A\bar{B})}{P(\bar{B})} \cdot P(\bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$, as required. $\square$

c. We can think of this as picking positions in the row for the balls to occupy. There are $C_1^{10} = \binom{10}{1}$ ways to pick one of the ten positions for the red ball, $C_2^9 = \binom{9}{2}$ ways to pick two of the remaining nine positions for the blue balls, $C_3^7 = \binom{7}{3}$ ways to pick three of the remaining seven positions for the green balls, and $C_4^4 = \binom{4}{4}$ ways to pick four of the remaining four positions for the yellow balls. Thus there are

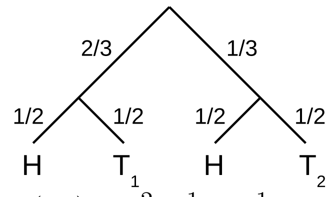
$$\binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{4} = \frac{10!}{9!1!} \cdot \frac{9!}{7!2!} \cdot \frac{7!}{4!3!} \cdot \frac{4!}{4!0!} = \frac{10!}{1!2!3!4!} = 525$$

ways to arrange the balls if balls of the same colour are indistinguishable. ■

3. Do either *one* (1) of **a** or **b**. *[8]*

a. Three fair coins have identical heads (H), and two of the three that were minted in the same year have identical tails (T_1), but the third was minted in a different year and has a distinct tail (T_2). One of the three coins is selected at random, tossed, and replaced; this is repeated for a total of two tosses. Let Z be the number of T_1s plus two times the number of T_2s that come up. Compute the expected value and standard deviation of Z.

b. Initially, jar I contains one blue and two red balls, jar II contains one red and two blue balls, and jar III is empty. Two balls are chosen randomly, without replacement, from each of jars I and II and placed in jar III. Let X be the number of red balls that end up in jar III. Compute the expected value and standard deviation of X.



SOLUTIONS. **a.** When selecting a coin there is a probability of $\frac{2}{3}$ of choosing one of the HT_1 coins and a $\frac{1}{3}$ chance of selecting the HT_2 coin. Since each coin is fair, the toss that follows has a probability of $\frac{1}{2}$ of coming up heads and $\frac{1}{2}$ of coming up tails. It follows that for any single experiment (select a coin and toss it), we have $P(H) = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}$, $P(T_1) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, and $P(T_2) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

The experiment of selecting a coin and tossing it is performed twice, so the sample space is $S = \{ HH, HT_1, HT_2, T_1H, T_2H, T_1T_1, T_1T_2, T_2T_1, T_2T_2 \}$; with the outcomes *not* being equally likely. $Z = \#T_1\text{s} + \#T_2\text{s}$ must be an integer which is at least 0 (the outcome HH) and at most 4 (the outcome T_2T_2). Its probability function is given by:

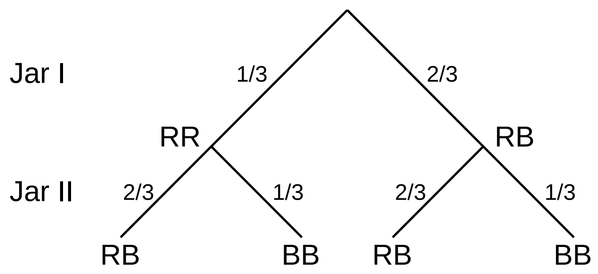
$$\begin{aligned}
p(0) &= P(Z=0) = P(\,HH\,) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
p(1) &= P(Z=1) = P(\,HT_1,\, T_1H\,) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{6} = \frac{1}{3} \\
p(2) &= P(Z=2) = P(\,T_1T_1,\, HT_2,\, T_2H\,) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{2} = \frac{5}{18} \\
p(3) &= P(Z=3) = P(\,T_1T_2,\, T_2T_1\,) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{9} \\
p(4) &= P(Z=4) = P(\,T_2T_2\,) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}
\end{aligned}$$

It follows that, by definition,

$$\begin{aligned}
E(Z) &= \sum_{z=0}^{4} zp(z) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{5}{18} + 3 \cdot \frac{1}{9} + 4 \cdot \frac{1}{36} = \frac{48}{36} = \frac{4}{3}\,, \\
V(Z) &= E\left(Z^2\right) - [E(Z)]^2 = \left(0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{5}{18} + 3^2 \cdot \frac{1}{9} + 4^2 \cdot \frac{1}{36}\right) - \left[\frac{4}{3}\right]^2 \\
&= \frac{26}{9} - \frac{16}{9} = \frac{10}{9}\,, \text{ and } \sigma(Z) = \sqrt{V(Z)} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}.\ \square
\end{aligned}$$

[Total = 30]

b. Note that the order in which balls are chosen from the jars doesn't matter here, so there are $C_2^3 = \binom{3}{2} = 3$ ways to choose two balls out of three without replacement out of each jar. Jar I has two red balls and one blue ball, so the three equally likely possibilities are choosing both red balls, choosing the blue ball and one of the red balls, and choosing the blue ball and the other red ball. We thus choose two red balls from jar I with probability $\frac{1}{3}$, and choose the blue ball and one of the red balls with probability $\frac{2}{3}$. A similar analysis tells us that we choose two blue balls from jar I with probability $\frac{1}{3}$, and choose the red ball and one of the blue balls with probability $\frac{2}{3}$.



It's not hard to see that choosing two balls from jar I will give us one or two red balls and choosing two balls from jar II will give us zero or one red balls, and so jar III ends up with one, two, or three red balls. It follows from all this that if X is the number of red balls that end up in in jar III, then its probability function must be given by:

$$p(1) = P(X=1) = P(\, RB\&BB\,) = \frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}$$
$$p(2) = P(X=2) = P(\, RR\&BB,\ RB\&RB\,) = \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3} = \frac{5}{9}$$
$$p(3) = P(X=3) = P(\, RR\&RB\,) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

It follows that, by definition,

$$E(X) = \sum_{x=1}^{3} xp(x) = 1\cdot\frac{2}{9} + 2\cdot\frac{5}{9} + 3\cdot\frac{2}{9} = \frac{18}{9} = 2\,,$$
$$V(X) = E\left(X^2\right) - [E(X)]^2 = \left(1^2\cdot\frac{2}{9} + 2^2\cdot\frac{5}{9} + 3^2\cdot\frac{2}{9}\right) - 2^2 = \frac{40}{9} - \frac{36}{9} = \frac{4}{9}\,,$$

and $\sigma(X) = \sqrt{V(X)} = \sqrt{\frac{4}{9}} = \frac{2}{3}$. ■