Mathematics 1550H – Introduction to probability
Trent University, Summer 2014

Solutions to Assignment #5
Can we have Tchebysheff's equality?

Recall from class and the textbook that the following is true:

Tchebysheff's Theorem. Suppose X is a random variable, discrete or continuous, with expected value $E(X) = \mu$ and standard deviation $\sigma(X) = \sigma$. Then, for any real number $k > 0$, $P\left(|X - \mu| < k\sigma^2\right) \geq 1 - \dfrac{1}{k^2}$.

Your task in this assignment, should you choose to accept it, will be to see if there are, under reasonable conditions, random variables X for which we actually get an equality instead, *i.e.* $P\left(|X - \mu| < k\sigma^2\right) = 1 - \dfrac{1}{k^2}$.

Do *one* (1) of problems **1** or **2** below. To keep things relatively simple, we will assume in both **1** and **2** that $\mu = 0$ and $\sigma = 1$, and also that X is symmetric about $\mu = 0$, *i.e.* that $P(0 \leq X \leq x) = P(-x \leq X \leq 0)$ for all $x \geq 0$.

1. Is there a discrete random variable X, satisfying the assumptions above, which takes on (not necessarily positive) integer values such that $P\left(|X| < k\right) = 1 - \dfrac{1}{k^2}$ for every positive integer k? If so, give an example; if not, explain why not. *[10]*

Solution. There is such a random variable. In fact, it turns out there is only possible probability function $p(k) = P(X = k)$ (where $k \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$) for such a variable, given the stated conditions it is supposed to satisfy:

- $p(0) = P\left(X = 0\right) = P\left(|X| < 1\right) = 1 - \frac{1}{1^2} = 1 - 1 = 0$.
- $p(-1) + p(0) + p(-1) = P\left(|X| < 2\right) = 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$, and since $p(0) = 0$ and $p(-1) = p(1)$ (because of the requirement that X be symmetric), it follows that $p(-1) = p(1) = \frac{3}{8}$.
- $p(-2) + p(-1) + p(0) + p(-1) + p(2) = P\left(|X| < 3\right) = 1 - \frac{1}{3^2} = 1 - \frac{1}{9} = \frac{8}{9}$, so $p(-2) + p(2) = P\left(|X| < 3\right) - P\left(|X| < 2\right) = \frac{8}{9} - \frac{3}{4} = \frac{5}{36}$, and so, since $p(-2) = p(2)$, $p(-2) = p(2) = \frac{5}{72}$.
- In general, if $k > 0$, $p(-k) + \cdots + p(k) = P\left(|X| < k+1\right) = 1 - \frac{1}{(k+1)^2}$, so $p(-k) + p(k) = P\left(|X| < k+1\right) - P\left(|X| < k\right) = \left[1 - \frac{1}{(k+1)^2}\right] - \left[1 - \frac{1}{k^2}\right] = \frac{1}{k^2} - \frac{1}{(k+1)^2} = \frac{(k+1)^2 - k^2}{k^2(k+1)^2} = \frac{k^2+2k+1-k^2}{k^2(k+1)^2} = \frac{2k+1}{k^2(k+1)^2}$. Since $p(-k) = p(k)$, it follows that $p(-k) = p(k) = \frac{2k+1}{2k^2(k+1)^2}$.

Obviously, we must have $p(x) = 0$ whenever $x \notin \mathbb{Z}$. Thus the probability function of a discrete random variable with the given properties must be:

$$p(x) = \begin{cases} \frac{2k+1}{2k^2(k+1)^2} & x = \pm k \text{ for an integer } k > 0 \\ 0 & x = 0 \text{ or } x \text{ is not an integer} \end{cases}$$

It remains to check that this is indeed a probability function. First, it is obvious from its definition that $p(x) \geq 0$ for all x. Second, we need to check that the total probability is 1, that is, that $\displaystyle\sum_x p(x) = 1$:

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} p(k) &= p(0) + \sum_{k=1}^{\infty} [p(-k) + p(k)] = 0 + \sum_{k=1}^{\infty} \left[\frac{2k+1}{2k^2(k+1)^2} + \frac{2k+1}{2k^2(k+1)^2} \right] \\
&= \sum_{k=1}^{\infty} 2 \cdot \frac{2k+1}{2k^2(k+1)^2} = \sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{\infty} \left[\frac{1}{k^2} - \frac{1}{(k+1)^2} \right] \\
&= \left[\frac{1}{1^2} - \frac{1}{(1+1)^2} \right] + \left[\frac{1}{2^2} - \frac{1}{(2+1)^2} \right] + \left[\frac{1}{3^2} - \frac{1}{(3+1)^2} \right] + \cdots \\
&= 1 + \left(-\frac{1}{4} + \frac{1}{4} \right) + \left(-\frac{1}{9} + \frac{1}{9} \right) + \left(-\frac{1}{16} + \frac{1}{16} \right) \cdots = 1 + 0 + 0 + \cdots = 1
\end{aligned}$$

Thus $p(x)$ satisfies the definition of a probability function. ■

2. Is there a continuous random variable X, satisfying the assumptions above, such that $P\left(|X| < k\right) = 1 - \dfrac{1}{k^2}$ for every positive real number k? If so, give an example; if not, explain why not. *[10]*

SOLUTION. There is no such continuous random variable. Suppose, for example, that $k = \dfrac{1}{2}$. Then we would have to have $P\left(|X| < \dfrac{1}{2}\right) = 1 - \dfrac{1}{\left(\frac{1}{2}\right)^2} = 1 - 4 = -3$, which would violate the requirement that probabilities be ≥ 0. ■

NOTE: Interesting how essentially the same requirements are possible to satisfy for a discrete random variable, but not a continuous one. One could make the problem more interesting in the continuous case by requiring that $P\left(|X| < k\right) = 1 - \dfrac{1}{k^2}$ only for real numbers $k \geq 1$. (You figure out what happens ... :-)