Mathematics 1550H – Introduction to probability

Trent University, Summer 2014

Assignment #4

Unexpected Value!?

The function $f(x) = \dfrac{1}{\pi\left(1+x^2\right)}$ is an unfortunate one for those who hoped continuous random variables would behave themselves. On the one hand:

1. Verify that $f(x)$ is a probability density function. *[5]*

Solution. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, $\pi\left(1+x^2\right) > 0$ for all $x \in \mathbb{R}$. It follows from this that $f(x) = \dfrac{1}{\pi\left(1+x^2\right)}$ is defined and continuous, and hence integrable, for all l $x \in \mathbb{R}$. Finally,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\pi\left(1+x^2\right)}\,dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\
&= \left[\lim_{t\to-\infty} \frac{1}{\pi}\int_{t}^{0} \frac{1}{1+x^2}\,dx\right] + \left[\lim_{s\to\infty} \frac{1}{\pi}\int_{0}^{s} \frac{1}{1+x^2}\,dx\right] \\
&= \left[\lim_{t\to-\infty} \frac{1}{\pi}\arctan(x)\Big|_{t}^{0}\right] + \left[\lim_{s\to\infty} \frac{1}{\pi}\arctan(x)\Big|_{0}^{s}\right] \\
&= \frac{1}{\pi}\lim_{t\to-\infty}\left(\arctan(0) - \arctan(t)\right) + \frac{1}{\pi}\lim_{s\to\infty}\left(\arctan(s) - \arctan(0)\right) \\
&= -\frac{1}{\pi}\lim_{t\to-\infty}\arctan(t) + \frac{1}{\pi}\lim_{s\to\infty}\arctan(s) \quad (\text{since } \arctan(0) = 0) \\
&= -\frac{1}{\pi}\left(-\frac{\pi}{2}\right) + \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2} + \frac{1}{2} = 1\,,
\end{aligned}$$

so $f(x) = \dfrac{1}{\pi\left(1+x^2\right)}$ is a continuous probability density function. □

2. Show that if the random variable X has $f(x)$ as its probability density function, then X does not have a well-defined expected value. *[5]*

Hint: Try computing $E(X)$ and see if you actually get a number ...

Solution. We'll follow the hint:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} \frac{x}{\pi\left(1+x^2\right)}\,dx = \frac{1}{\pi}\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \frac{1}{\pi}\int_{0}^{\infty} \frac{x}{1+x^2}\,dx \\
&= \left[\lim_{t\to-\infty} \frac{1}{\pi}\int_{t}^{0} \frac{x}{1+x^2}\,dx\right] + \left[\lim_{s\to\infty} \frac{1}{\pi}\int_{0}^{s} \frac{x}{1+x^2}\,dx\right]
\end{aligned}$$

We'll substitute $u = 1+x^2$, so $du = 2x\,dx$ and thus $\frac{1}{2}\,du = x\,dx$, and $\begin{smallmatrix} x & t & 0 & s \\ u & 1+t^2 & 1 & 1+s^2 \end{smallmatrix}$.

$$= \left[\lim_{t\to-\infty} \frac{1}{\pi}\int_{1+t^2}^{1} \frac{1}{u}\cdot\frac{1}{2}\,du\right] + \left[\lim_{s\to\infty} \frac{1}{\pi}\int_{1}^{1+s^2} \frac{1}{u}\cdot\frac{1}{2}\,du\right]$$

$$= \left[\frac{1}{2\pi}\lim_{t\to-\infty}\ln(u)\Big|_{1+t^2}^{1}\right] + \left[\frac{1}{2\pi}\lim_{s\to\infty}\ln(u)\Big|_{1}^{1+s^2}\right]$$

$$= \frac{1}{2\pi}\left[\lim_{t\to-\infty}\left(\ln(1)-\ln\left(1+t^2\right)\right)\right] + \frac{1}{2\pi}\left[\lim_{s\to\infty}\left(\ln\left(1+s^2\right)-\ln(1)\right)\right]$$

$$= \frac{1}{2\pi}\left[-\lim_{t\to-\infty}\ln\left(1+t^2\right)\right] + \frac{1}{2\pi}\left[\lim_{s\to\infty}\ln\left(1+s^2\right)\right] \qquad \text{(Since } \ln(1)=0.\text{)}$$

At this point we run into an insuperable problem: as $t \to -\infty$, $\left(1+t^2\right) \to \infty$, so $\lim_{t\to-\infty}\ln\left(1+t^2\right) = \infty$, and as $s \to \infty$, $\left(1+s^2\right) \to \infty$, so $\lim_{s\to\infty}\ln\left(1+s^2\right) = \infty$, too. That is, we do not get a real number for $E(X)$, just a difference of infinities, which is indeterminate. Hence $E(X)$ is not well-defined. $\square$

Bonus. Find a function $g(x)$ such that a random variable X which has $g(x)$ as its probability density function does have a well-defined expected value $E(X)$, but does not have a well-defined variance $V(X)$. *[2]*

Solution. Try computing $E(X)$ and $V(X)$ if X has the probability density function $g(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$, and see what happens ... $\blacksquare$