Mathematics 1550H – Introduction to probability
Trent University, Summer 2014
Solutions to Assignment #3
Simulations

A common problem in probability courses is to figure out how to use a possibly biased coin to simulate a fair coin. (See, for example, Assignment #3 from the Summer 2013 edition of MATH 1550H ... :-) The basic idea behind the usual solution to this problem can be used to do a little more.

1. Suppose you are stuck on a desert island with nothing but a fair coin, but you feel the urge to play a game that requires you to roll a fair (standard six-sided) die. How can you simulate rolling a die using the coin you do have? *[3]*

Solution. Toss the coin three times (with order) and interpret the results as follows:

$$\begin{array}{cccccccc} HHH & HHT & HTH & HTT & THH & THT & TTH & TTT \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & \text{start over} & \text{start over} \end{array}$$

Since any of the eight outcomes of tossing the coin three times is as likely as any other, the coin being fair, you are as likely to get an outcome corresponding to a given face on the simulated die as you are an outcome corresponding to any other given face on the simulated die. The fact that two of the eight outcomes prompt you to try again is an annoyance, but does not affect the relative probabilities of the simulated die faces. Theoretically one could have an infinite string of start overs, but this possibility has probability zero ... □

2. Suppose you are (still!) stuck on the desert island, but you get the urge to play a game that requires tossing a biased coin, with $P(H) = \frac{13}{20} = 0.65$ and $P(T) = \frac{7}{20} = 0.35$. How can you simulate tossing such an unfair coin using the fair coin you do have? *[3]*

Solution. The least power of 2 greater than or equal to 20 is $2^5 = 32$. Toss the fair coin five times and interpret the result as follows, where **H** and **T** are used to denote the faces of the simulated biased coin:

$$\mathbf{H} \leftarrow \begin{cases} HHHHH, HHHHT, HHHTH, HHHTT, HHTHH, HHTHT, HHTTH, HHTTT, \\ HTHHH, HTHHT, HTHTH, HTHTT, HTTHH \end{cases}$$

$$\mathbf{T} \leftarrow \{\, HTTHT, HTTTH, HTTTT, THHHH, THHHT, THHTH, THHTT$$

$$\begin{matrix}\text{start}\\ \text{over}\end{matrix} \leftarrow \begin{cases} THTHH, THTHT, THTTH, THTTT, TTHHH, TTHHT, TTHTH, TTHTT, \\ TTTHH, TTTHT, TTTTH, TTTTT \end{cases}$$

That is, 13 of the 32 sequences of five tosses of the fair coin correspond to a head (**H**) of the simulated biased coin, 7 of the 32 sequences of five tosses of the fair coin correspond to a tail (**T**) of the simulated biased coin, and the remaining 12 are a signal to try again. Since the actual coin being used is fair and there are $13 + 7 = 20$ decisive outcomes which do not require one to start over, the probability of (eventually) getting a **H** is $\frac{13}{20}$, and the probability of (eventually) getting a **T** is $\frac{7}{20}$, as desired. Again, one could theoretically have an infinite string of start overs, but this possibility has probability zero ... □

3. In general, suppose (a complete description of) a random process with a finite sample space S and probabilities, not necessarily equal, for all the outcomes in S are given. Can one simulate this process by tossing a fair coin? Just when and why is it possible to do so, and, when one can, how? *[4]*

Solution. Suppose a complete description of a random process with a finite sample space S and probabilities, not necessarily equal, for all the outcomes in S is given. It can be simulated by tossing a fair coin repeatedly so long as the probabilities for all the outcomes in S are rational numbers (*i.e.* ratios of integers):

Suppose $S = \{ s_1, s_2, \dots, s_n \}$ and $P(s_k) = \dfrac{a_k}{b_k}$ for each k with $1 \le k \le n$, where a_k and b_k are positive integers for each k. Let d be the least common multiple of all of b_1, b_2, $\dots$, b_n; then for each k we can write the probability of outcome s_k as $P(s_k) = \dfrac{a_k}{b_k} = \dfrac{c_k}{d}$ for some positive integer c_k. Now let p be the least integer such that $2^p \ge d$; then there are 2^p possible sequences of p tosses of a fair coin, all equally likely. Arrange these sequences in some order (alphabetical order was used in the solutions to **1** and **2** above), but any order will do), and correspond the first c_1 sequences to s_1, the next c_2 sequences to s_2, the next c_3 sequences after that to s_3, and so on, with $2^p - d$ sequences being left over. Toss the fair coin p times; if the sequence obtained corresponds to one of the outcomes s_k, you're done, if it is one of those left over, toss the coin p times again. (Repeat as necessary.)

By its construction, this process will (eventually) give you one of the sequences corresponding to s_k with probability $P(s_k) = \dfrac{c_k}{d} = \dfrac{a_k}{b_k}$. (There is, of course, the theoretical risk of getting leftover sequences forever, but this has probability zero ...)

Note that this kind of trick cannot be pulled off if some of the outcomes in S have probabilities that are not rational, such as $\frac{1}{\sqrt{2}}$. ■