Mathematics 1550H – Introduction to probability
Trent University, Summer 2014

Solutions to Assignment #2
Deals

A hand of five cards is dealt – randomly, one card at a time, without replacement – from each of two standard decks of 52 cards.

1. What is the probability that the two hands have exactly one card in common? *[3]*

Solution. The underlying sample space S consists of all pairs of hands, each drawn randomly, in order without replacement, from a separate deck. This means that there are $n(S) = P_5^{52} \cdot P_5^{52} = \frac{52!}{(52-5)!} \cdot \frac{52!}{(52-5)!} = (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)$ possible outcomes (*i.e.* pairs of hands) in S, all equally likely.

Let A be the event that the two hands have exactly one card in common. To count the number of outcomes in A, we need to combine the numbers of ways

- to pick the card they have in common, $P_1^{52} = 52$,
- to pick the position that card occupies in the first hand, $P_1^5 = 5$,
- to pick the position that card occupies in the second hand, $P_1^5 = 5$,
- to pick the remaining four cards in the first hand, $P_4^{51} = 51 \cdot 50 \cdot 49 \cdot 48$, and
- to pick the remaining four cards in the second hand, which must be different from the remaining four in the first hand, $P_4^{47} = 47 \cdot 46 \cdot 45 \cdot 44$.

Since these choices are independent, except for the points noted above, there are $n(A) = 52 \cdot 5 \cdot 5 \cdot (51 \cdot 50 \cdot 49 \cdot 48) \cdot (47 \cdot 46 \cdot 45 \cdot 44)$ outcomes in A.

It follows that

$$\begin{aligned} P(A) = \frac{n(A)}{n(S)} &= \frac{52 \cdot 5 \cdot 5 \cdot (51 \cdot 50 \cdot 49 \cdot 48) \cdot (47 \cdot 46 \cdot 45 \cdot 44)}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)} \\ &= \frac{5 \cdot 5 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{59455}{173264} \approx 0.3433146874\,. \qquad \blacksquare \end{aligned}$$

2. What is the probability that the two hands have at least one card in common? *[3]*

Solution. Let B be the event that the two hands have at least one card in common. Rather than work out $P(B)$ directly, we will work out $P\left(\bar{B}\right)$, the probability that the two hands have no cards in common, first.

$\bar{B}$ consists of all pairs of hands in which the two hands have no cards in common. There are $P_5^{52} = \frac{52!}{(52-5)!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ ways to pick the first hand. This leaves only $52 - 5 = 47$ cards which may be used for the second hand, so there are $P_5^{47} = \frac{47!}{(47-5)!} = 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43$ ways to pick the second hand. Thus $n\left(\bar{B}\right) = P_5^{52} \cdot P_5^{47} = (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (47 \cdot 46 \cdot 45 \cdot 44 \cdot 43)$, and it follows that

$$\begin{aligned} P\left(\bar{B}\right) = \frac{n\left(\bar{B}\right)}{n(S)} &= \frac{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (47 \cdot 46 \cdot 45 \cdot 44 \cdot 43)}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)} \\ &= \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{511313}{866320} \approx 0.5902126235\,. \end{aligned}$$

Hence $P(B) = 1 - P\left(\bar{B}\right) = \frac{355007}{866320} \approx 0.4097873765$. $\blacksquare$

3. Determine whether the events of "the two hands have at least one card in common" and "the first hand is all $\heartsuit$s" are independent or not. *[4]*

Solution. Let B be the event that the two hands have at least one card in common and C be the event that the first hand is all $\heartsuit$s. By definition, B and C are independent if $P(BC) = P(B) \cdot P(C)$. We know from the answer to **2** above that $P(B) = \frac{355007}{866320} \approx 0.4097873765$, so we will compute $P(C)$ and $P(BC)$ and then check to see if the defining equation holds.

C consists of all pairs of hands in which the first hand is all $\heartsuit$s. Since there are thirteen cards in each suite, there are $P_5^{13} = \frac{13!}{(13-5)!} = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$ ways to pick the first hand, and $P_5^{52} = \frac{52!}{(52-5)!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ ways to pick the second hand (as there are no restrictions on the second hand here). Thus $n(C) = P_5^{13} \cdot P_5^{52} = (13 \cdot 12 \cdot 11 \cdot 10 \cdot 9) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)$, and so it follows that

$$\begin{aligned} P(C) = \frac{n(C)}{n(S)} &= \frac{(13 \cdot 12 \cdot 11 \cdot 10 \cdot 9) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)} \\ &= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{33}{66640} \approx 0.0004951981\,. \end{aligned}$$

Rather than work out $P(BC)$ directly, we will work out $P\left(\bar{B}C\right)$, the probability that the two hands have no cards in common and that the first hand is all $\heartsuit$s, first. as above, there are $P_5^{13} = \frac{13!}{(13-5)!} = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$ ways to pick the first hand, and, as in the solution to **2**, this leaves only $52-5 = 47$ cards which may be used for the second hand, so there are $P_5^{47} = \frac{47!}{(47-5)!} = 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43$ ways to pick the second hand. Thus $n\left(\bar{B}C\right) = P_5^{13} \cdot P_5^{47} = (13 \cdot 12 \cdot 11 \cdot 10 \cdot 9) \cdot (47 \cdot 46 \cdot 45 \cdot 44 \cdot 43)$, and it follows that

$$\begin{aligned} P\left(\bar{B}C\right) = \frac{n\left(\bar{B}C\right)}{n(S)} &= \frac{(13 \cdot 12 \cdot 11 \cdot 10 \cdot 9) \cdot (47 \cdot 46 \cdot 45 \cdot 44 \cdot 43)}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)} \\ &= \frac{16873329}{57731564800} \approx 0.0002922722\,. \end{aligned}$$

Since BC and $\bar{B}C$ partition C, we have $P(C) = P(BC) + P\left(\bar{B}C\right)$, and so $P(BC) = P(C) - P\left(\bar{B}C\right) \approx 0.0004951981 - 0.0002922722 = 0.0002029259$.

As $P(B) \cdot P(C) \approx 0.4097873765 \cdot 0.0004951981 \approx 0.0002029259 \approx P(BC)$, it is thus *very* likely that B and C are independent events. (To be *really* sure, we should work everything out *exactly*, but the fractional arithmetic is just a bit painful ... :-) ■