Mathematics 1550H – Introduction to probability
Trent University, Summer 2014
Solutions to the Quizzes

Quiz #1. Thursday, 25 June, 2014. *[15 minutes]*

A standard six-sided die is modified to have the number 1 appear on one face, 2 on two faces, and 3 on the remaining three faces. The die is otherwise fair.

0. If the die is thrown once, what is the probability of each number coming up? *[1]*

Now suppose the modified die is thrown twice, producing an ordered pair (a, b) of numbers, where a is the result of the first throw and b is the result of the second throw.

1. What is the sample space for this experiment? *[1]*
2. Determine the probability $P(F)$ of the event F that exactly one of the tosses gives 2. *[1.5]*
3. Determine the probability $P(E)$ of the event E that the sum of the two tosses is even. *[1.5]*

Solutions. 1. There are six faces, each of which is equally likely to come up when the die is tossed, since it is supposed to be fair (apart from how the faces are marked). Since 1 appears on just one of the six faces, the probability that 1 will come up is $P(1) = \frac{1}{6}$; since 2 appears on exactly two of the six faces, the probability that 2 will come up is $P(2) = \frac{2}{6} = \frac{1}{3}$; and since 3 appears on exactly three of the six faces, the probability that 3 will come up is $P(3) = \frac{3}{6} = \frac{1}{2}$. □

2. The sample space consists of all possible outcomes for the experiment, that is, of all ordered pairs (a, b) such that each of a and b is 1, 2, or 3. For the notationally inclined, the sample space is

$$\begin{aligned} S &= \{\, (a, b) \mid a = 1,\ 2, \text{ or } 3, \text{ and } b = 1,\ 2, \text{ or } 3 \,\} \\ &= \{\, (1, 1),\ (1, 2),\ (1, 3),\ (2, 1),\ (2, 2),\ (2, 3),\ (3, 1),\ (3, 2),\ (3, 3) \,\} \,. \end{aligned}$$

Note that S includes nine possible outcomes, which are *not* equally likely. For example $P(1, 1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ but $P(3, 2) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. □

3. We compute the probability of the event by adding up the probabilities of the outcomes in that event:

$$\begin{aligned} P(F) &= P\,(\text{Exactly one of the two tosses gives 2.}) = P\left(\{\, (1, 2),\ (2, 1),\ (2, 3),\ (3, 2) \,\}\right) \\ &= P(1, 2) + P(2, 1) + P(2, 3) + P(3, 2) = \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{18} + \frac{1}{18} + \frac{1}{6} + \frac{1}{6} = \frac{1}{18} + \frac{1}{18} + \frac{3}{18} + \frac{3}{18} = \frac{8}{18} = \frac{4}{9} \qquad \square \end{aligned}$$

4. *(With brutal directness.)* Again, we compute the probability of the event by adding up the probabilities of the outcomes in that event. Note that the sum of two integers is even if the two integers are both even or both odd; if one is even and the other odd, their sum will be odd.

$$\begin{aligned} P(E) &= P\,(\text{The sum of the two tosses is even.}) = P\left(\{\, (1, 1),\ (1, 3),\ (2, 2),\ (3, 1),\ (3, 3) \,\}\right) \\ &= P(1, 1) + P(1, 3) + P(2, 2) + P(3, 1) + P(3, 3) = \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{36} + \frac{1}{12} + \frac{1}{9} + \frac{1}{12} + \frac{1}{4} = \frac{1}{36} + \frac{3}{36} + \frac{4}{36} + \frac{3}{36} + \frac{9}{36} = \frac{20}{36} = \frac{5}{9} \qquad \square \end{aligned}$$

4. *(With a little cleverness.)* Since 2 is the only even number among 1, 2, and 3, and the sum of two integers is even exactly when the two integers are both even or both odd, we get that

$$E = \text{The sum of the two tosses is even} = \text{2 does not occur exactly once} = \overline{F}\,.$$

It follows that $P(E) = P\left(\overline{F}\right) = 1 - P(F) = 1 - \frac{4}{9} = \frac{5}{9}$. ■

Quiz #2. Tuesday, 30 June, 2014. *[15 minutes]*

A hand of five cards is dealt out face up, one card at a time and without replacement, from a standard 52-card deck (A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2 of each of four suites, $\heartsuit$, $\diamondsuit$, $\clubsuit$, and $\spadesuit$).

1. How many such hands are there? *[1]*
2. How many hands are there in which all five cards are from the same suite? *[1]*

Wanda Maximoff enchants the deck so that every card drawn from it is instantly replaced with a perfect copy and the deck magically reshuffled before the next card is drawn. A hand of five cards is dealt out face down, one card at a time, from the enchanted deck, but the order the cards were drawn in is lost when they are turned face up, so order does not matter.

3. How many possible such hands are there? *[2]*
4. How many hands are there in which all five cards are from the same suite? *[1]*

Bonus. Who is Wanda Maximoff? *[0.1]*

SOLUTIONS. 1. There are 52 possibilities for the first card to be dealt, 51 for the second (the first one dealt no longer being in the deck), 50 for the third, and so on. Thus there are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{(52-5)!} = 311,875,200$ possible hands. [It's OK not to work these numbers out on quizzes!] $\square$

2. There are four suites and thirteen cards in each suite. For each suite, there are $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = \frac{13!}{(13-5)!} = 154,440$ five-card ordered hands with all cards from that suite. It follows that there are $4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 4 \cdot 154,440 = 617,760$ hands where all five cards from the same suite. $\square$

3. Since the order the cards were chosen in is lost, we are effectively choosing five cards from 52 cards with replacement and without order, so there are $\binom{52+5-1}{5} = \binom{56}{5} = 3,819,816$ ways to do it, *i.e.* there are $\binom{52+5-1}{5} = \binom{56}{5} = 3,819,816$ possible hands. $\square$

4. For each suite, there are $\binom{13+5-1}{5} = \binom{17}{5} = 1287$ possible hands, *i.e.* collections of five cards chosen with replacement and without order. As there are four suites, it follows that there are $4 \cdot \binom{13+5-1}{5} = 4 \cdot \binom{17}{5} = 4 \cdot 1287 = 5148$ hands, chosen with replacement and without order, in which all the cards are from the same suite. $\square$

Bonus. Wanda Maximoff, also known as the Scarlet Witch, is a mutant and magic user in the Marvel Comics universe. Her mutant and magical powers involve controlling probabilities. $\blacksquare$

Quiz #3. Thursday, 2 July, 2014. *[15 minutes]*

1. Max is given fireworks for the holiday, seven distinct rockets and nine distinct Roman candles. [That is, you can tell different rockets, or different Roman candles, apart.] He proceeds to set them all off, one after another, in random order. If the fourth and fifth fireworks were both rockets, what is the probability that first three were all Roman candles? *[5]*

SOLUTION. Let A be the event that the first three fireworks are all Roman candles, and B be the event that the fourth and fifth fireworks are rockets. We need to compute the conditional probability $P(A|B) = \frac{P(AB)}{P(B)}$.

First, there are sixteen $(16 = 7 + 9)$ fireworks in total, and our sample space S consists of all possible arrangements of them. Since they are all distinct [that is, you can tell them all apart], order totally matters, and so there are $16!$ possible arrangements in S, each equally likely to be randomly chosen.

Second, there are $7 \cdot 6 = \frac{7!}{(7-2)!} = \frac{7!}{5!} = 42$ ways to choose rockets to be the fourth and fifth fireworks. Having done so, there are $(16-2)! = 14!$ ways to arrange the remaining five rockets and nine Roman candles in the remaining positions. Thus the event B includes $42 \cdot 14!$ arrangements. Since all arrangements are equally likely, it follows that

$$P(B) = \frac{n(B)}{n(S)} = \frac{42 \cdot 14!}{16!} = \frac{42}{16 \cdot 15} = \frac{42}{240} = \frac{7}{40}\,.$$

Third, there are $9 \cdot 8 \cdot 7 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 504$ ways to pick Roman candles to be the first three fireworks, and then there are $7 \cdot 6 = \frac{7!}{(7-2)!} = \frac{7!}{5!} = 42$ ways to choose rockets to be the fourth and fifth fireworks. Having done so, there are $(16-5)! = 11!$ ways to arrange the remaining five rockets and six Roman candles in the remaining positions. Thus the intersection of the events A and B includes $504 \cdot 42 \cdot 11!$ arrangements. Since all arrangements are equally likely, it follows that

$$P(AB) = \frac{n(AB)}{n(S)} = \frac{504 \cdot 42 \cdot 11!}{16!} = 21168 \cdot \frac{11!}{16!} = \frac{21168}{524160} = \frac{21}{520}\,.$$

Finally, the probability that the first three fireworks were all Roman candles, given that the fourth and and fifth were rockets, is

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{21}{520}}{\frac{7}{40}} = \frac{21}{520} \cdot \frac{40}{7} = \frac{3}{13}\,.$$

[It's OK not to work out all the numbers and simplify all the fractions here, as this takes a fair bit of time and/or a calculator.] Please note that A and B are not independent events. For example, if A happens and Roman candles occur as the first three fireworks, that increases the relative proportion of rockets among the remaining fireworks, and hence increases the likelihood that the fourth and fifth fireworks will be rockets, *i.e.* that B occurs. ■

Quiz #4. Monday, 7 July, 2014. *[15 minutes]*

A jar contains the three balls inscribed with the numbers 1, 2, and 3, respectively. Two balls are randomly chosen, in order with replacement, from the jar. Let the random variable X be the sum of the numbers inscribed on the chosen balls.

1. Find the probability function of X. *[3]*
2. Compute the expected value, $E(X)$, of X. *[2]*

SOLUTIONS. 1. First, the sample space consists of all ordered pairs of the numbers 1, 2, and 3:

$$S = \{\,(1,1),\ (1,2),\ (1,3),\ (2,1),\ (2,2),\ (2,3),\ (3,1),\ (3,2),\ (3,3)\,\}$$

There are nine possible outcomes in S, all equally likely.

Second, it's not hard to see that the possible values of X are $2 = 1+1$, $3 = 1+2 = 2+1$, $4 = 1+3 = 2+2 = 3+1$, $5 = 2+3 = 3+2$, and $6 = 3+3$.

It follows that the probability function of X is given by:

$$\begin{aligned}
p(2) &= P(X=2) = P\,(\,(1,1)\,) = \frac{1}{9}\\
p(3) &= P(X=3) = P\,(\,(1,2),\ (2,1)\,) = \frac{2}{9}\\
p(4) &= P(X=4) = P\,(\,(1,3),\ (2,2),\ (3,1)\,) = \frac{3}{9} = \frac{1}{3}\\
p(5) &= P(X=5) = P\,(\,(2,3),\ (3,2)\,) = \frac{2}{9}\\
p(6) &= P(X=6) = P\,(\,(3,3)\,) = \frac{1}{9}
\end{aligned}$$

Note that if $x \neq 2$, 3, 4, 5, or 6 then $p(x) = P(X=x) = 0$, since X can only take on the values 2, 3, 4, 5, or 6. $\square$

2. Using the definition of expected value for a discrete random variable:

$$\begin{aligned}
E(X) = \sum_{\text{possible } x} xp(x) &= 2p(2) + 3p(3) + 4p(4) + 5p(5) + 6p(6)\\
&= 2\cdot\frac{1}{9} + 3\cdot\frac{2}{9} + 4\cdot\frac{3}{9} + 5\cdot\frac{2}{9} + 6\cdot\frac{1}{9}\\
&= \frac{2+6+12+10+6}{9} = \frac{36}{9} = 4 \qquad \blacksquare
\end{aligned}$$

Quiz #5. Wednesday, 9 July, 2014. *[15 minutes]*

The roll of a fair die is declared to be a success if 3 or 6 comes up, and a failure otherwise. Let X be the number of successes in twelve (12) rolls of the die.

1. What kind of probability distribution does X have? *[1]*
2. Give the probability function of X. *[2]*
3. Compute the expected value, $E(X)$, and standard deviation, $\sigma(X)$, of X. *[2]*

Solutions. 1. Note that the probability of success on a single roll is $\frac{2}{6} = \frac{1}{3}$. X has a binomial distribution with $n = 12$ and $p = \frac{1}{3}$. You can think of the process as flipping a biased coin, with has a probability of $\frac{1}{3}$ of coming up heads on a given toss, twelve times and counting the number of heads that occur. $\square$

2. The probability function of a random variable with a binomial distribution with $n = 12$ and $p = \frac{1}{3}$ is

$$\begin{aligned} p(x) &= \binom{n}{x} p^x (1-p)^{n-x} = \binom{12}{x} \left(\frac{1}{3}\right)^x \left(1 - \frac{1}{3}\right)^{12-x} = \binom{12}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{12-x} \\ &= \frac{12!}{(12-x)!x!} \cdot \frac{2^{12-x}}{3^x 3^{12-x}} = \frac{12!\, 2^{12-x}}{(12-x)!x!\, 3^{12}} \quad \text{[If you really want to gild the lily!]} \end{aligned}$$

for $x = 0, 1, 2, \ldots, 12$, and $p(x) = 0$ otherwise. $\square$

3. The expected value of a random variable with a binomial distribution with $n = 12$ and $p = \frac{1}{3}$ is $E(X) = np = 12 \cdot \frac{1}{3} = 4$, and its standard deviation is $\sigma(X) = \sqrt{np(1-p)} = \sqrt{12 \cdot \frac{1}{3} \cdot \left(1 - \frac{1}{3}\right)} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$. $\blacksquare$

Note. One could, of course, work all of this out directly from the definitions, but it is *so* much easier to just memorize or look up the formulas ...

Quiz #6. Wednesday, 16 July, 2014. *[15 minutes]*

Suppose the continuous random variable X has $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$ as its probability density function.

1. Verify that $f(x)$ is indeed a probability density function. *[3]*
2. Compute the expected value $E(X)$ of X. *[2]*

Solutions. 1. First, $f(x)$ is defined for all $x \in \mathbb{R}$, and since it it is continuous at almost every point, it is integrable. Second, it is obvious from its definition that $f(x) \ge 0$ for all x. Third,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 1\, dx + \int_{1}^{\infty} 0\, dx = 0 + x|_0^1 + 0 = 1 - 0 = 1\,.$$

Thus $f(x)$ satisfies all three conditions for a probability density function. $\square$

2. Using the definition of the expected value of a random variable,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{0} 0x\, dx + \int_{0}^{1} 1x\, dx + \int_{1}^{\infty} 0x\, dx = 0 + \left.\frac{x^2}{2}\right|_0^1 + 0 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}\,. \quad \blacksquare$$

Quiz #7. Monday, 21 July, 2014. *[15 minutes]*

Suppose X is a continuous exponential random variable with parameter $\theta > 0$, so it has density function $f(x) = \begin{cases} \frac{1}{\theta}e^{-x/\theta} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

1. Find the *median* of X, *i.e.* the m such that $P(X \leq m) = P(X > m) = \dfrac{1}{2}$, in terms of θ. *[5]*

Solution. So long as $m \geq 0$ (and it's obvious it has to be > 0, isn't it?):

$$P(X \leq m) = \int_0^m f(x)\,dx = \int_0^m \frac{1}{\theta}e^{-x/\theta}\,dx = \int_0^{-m/\theta} e^u\ (-1)\ du$$

(Substituting $u = -x/\theta$, so $du = -\frac{1}{\theta}\,dx$ and thus $\frac{1}{\theta}\,dx = (-1)\,du$, and also $\begin{matrix} x & 0 & m \\ u & 0 & -m/\theta \end{matrix}$.)

$$= -e^u\big|_0^{-m/\theta} = \left[-e^{-m/\theta}\right] - \left[-e^0\right] = -e^{-m/\theta} + 1 = 1 - e^{-m/\theta}$$

We thus need to solve the equation $1 - e^{-m/\theta} = P(X \leq m) = \frac{1}{2}$ for m:

$$1 - e^{-m/\theta} = \frac{1}{2} \implies e^{-m/\theta} = \frac{1}{2} \implies -\frac{m}{\theta} = \ln\left(\frac{1}{2}\right) \implies m = -\theta\ln\left(\frac{1}{2}\right)$$
$$\implies m = -\theta\ln\left(2^{-1}\right) \implies m = -\theta(-1)\ln(2) = \theta\ln(2) \qquad \blacksquare$$

Quiz #8. Wednesday, 23 July, 2014. *[10 minutes]*

1. Suppose X is a continuous random variable that has a normal distribution with parameters $\mu = 1$ and $\sigma = 2$. Use your table for the standard normal distribution to estimate the probability that $X < 0$. *[5]*

Solution. By the definition of the normal distribution,

$$P(X < 0) = \int_{-\infty}^0 \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx = \int_{-\infty}^0 \frac{1}{2\sqrt{2\pi}}e^{-(x-1)^2/2\cdot 2^2}\,dx$$

We can recast this in terms of the standard normal distribution using the substitution $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-1}{2}$, so $dz = \dfrac{1}{2}\,dx$ and $\begin{matrix} x & -\infty & 0 \\ z & -\infty & -1/2 \end{matrix}$, so

$$P(X < 0) = \int_{-\infty}^0 \frac{1}{2\sqrt{2\pi}}e^{-(x-1)^2/2\cdot 2^2}\,dx = \int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz\,.$$

Note that the substituted integral gives $P\left(Z < -1/2\right)$, where Z is the random variable $Z = \dfrac{X-1}{2}$.

Looking up our table, we see that it gives values for integrals of the form $\displaystyle\int_0^a \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz$, but a little algebra exploiting the properties of integrals and the facts that $e^{-z^2/2} = e^{-(-z)^2/2}$ and $\displaystyle\int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz = \frac{1}{2}$ lets us work around this:

$$\int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz - \int_{-1/2}^0 \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz = \frac{1}{2} - \int_0^{1/2} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz$$

Thus $P(X < 0) = P(Z < -0.5) = 0.5 - 0.1915 = 0.3085$. $\blacksquare$

Quiz #9. Monday, 28 July, 2014. *[12 minutes]*

Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	0	1
1	0.1	0.2
2	0.3	0.1
3	0.1	0.2

1. Compute $\text{cov}(X,Y)$, the covariance between X and Y. *[4]*
2. Determine whether X and Y are independent or not. *[1]*

SOLUTIONS. 1. To compute the covariance between X and Y, $\text{cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$, we need to compute the expected values of X, Y, and XY.

$$\begin{aligned} E(X) &= 1P(X=1) + 2P(X=2) + 3P(X=3) = 1(0.1+0.2) + 2(0.3+0.1) + 3(0.1+0.2) \\ &= 0.3 + 0.8 + 0.9 = 2 \\ E(Y) &= 0P(Y=0) + 1P(Y=1) = 0(0.1+0.3+0.1) + 1(0.2+0.1+0.2) = 0 + 0.5 = 0.5 \\ E(XY) &= 0P(XY=0) + 1P(XY=1) + 2P(XY=2) + 3P(XY=3) \\ &= 0(0.1+0.3+0.1) + 1 \cdot 0.2 + 2 \cdot 0.1 + 3 \cdot 0.2 = 0 + 0.2 + 0.2 + 0.6 = 1 \end{aligned}$$

It follows that $\text{cov}(X,Y) = E(XY) - E(X) \cdot E(Y) = 1 - 2 \cdot 0.5 = 1 - 1 = 0$. □

2. Since $\text{cov}(X,Y) = 0$, X and Y *might* be independent. However, they turn out to fail the definition of independence, which requires that $P(X=x, Y=y) = P(X=x)P(Y=y)$ for all real numbers x and y. For example, $P(X=2) = 0.3 + 0.1 = 0.4$ and $P(Y=0) = 0.1 + 0.3 + 0.1 = 0.5$, but $P(X=2, Y=0) = 0.3 \neq 0.2 = 0.4 \cdot 0.5 = P(X=2)P(Y=0)$. ■

Quiz #10. Wednesday, 30 July, 2014. *[10 minutes]*

Suppose X and Y are random variables such that $E(X) = 1.5$, $V(X) = 0.25$, $E(Y) = -0.2$, $V(Y) = 0.96$, and $\text{cov}(X,Y) = -0.8$.

1. Let $U = 2X - Y$. Compute $E(U)$ and $V(U)$. *[5]*

SOLUTION. $E(U) = E(2X - Y) = 2E(X) - E(Y) = 2 \cdot 1.5 - (-0.2) = 3.2$ and

$$\begin{aligned} V(U) &= V(2X - Y) = V(2X + (-1)Y) = 2^2 V(X) + (-1)^2 V(Y) + 2 \cdot 2 \cdot (-1)\text{cov}(X,Y) \\ &= 4 \cdot 0.25 + 1 \cdot 0.96 - 4 \cdot (-0.8) = 1 + 0.96 + 3.2 = 5.16\,. \quad ■ \end{aligned}$$