Mathematics 1550H – Introduction to probability

Trent University, Summer 2014

Solutions to the Final Examination

Friday, 6 August, 2014

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do both of parts ♡ and ♢, and, if you wish, part ♣. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; $\leq 10^{10}!$ neurons.

Part ♡. Do all of **1–5**. *[Subtotal = 70/100]*

1. A fair standard six-sided die is rolled four times. Let X be the number of times that odd-numbered faces come up, and let Y be the sum of the odd-numbered faces that come up.

a. What is the probability function of X? *[5]*

b. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. *[5]*

c. Without computing it from scratch, what should $E(Y)$ be? Explain why. *[5]*

Solutions. **a.** On any one roll of the die the probability that an odd-numbered face comes up is $\frac{1}{2}$, because three of the six equally likely faces have odd numbers. Thus X counts the number of successes in $n = 4$ rolls where the probability of success on each roll is $p = \frac{1}{2}$, which means X has a binomial distribution with these parameters. It follows that the probability function of X is:

$$p(x) = \begin{cases} \binom{4}{x}\left(\frac{1}{2}\right)^x\left(1-\frac{1}{2}\right)^{4-x} & x = 0,\ 1,\ 2,\ 3,\ 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{4}{x}\frac{1}{16} & x = 0,\ 1,\ 2,\ 3,\ 4 \\ 0 & \text{otherwise} \end{cases} \qquad \square$$

b. Using the definitions:

$$\begin{aligned} E(X) &= \sum_x x p(x) = 0\binom{4}{0}\frac{1}{16} + 1\binom{4}{1}\frac{1}{16} + 2\binom{4}{2}\frac{1}{16} + 3\binom{4}{3}\frac{1}{16} + 4\binom{4}{4}\frac{1}{16} \\ &= \frac{0}{16} + \frac{4}{16} + \frac{12}{16}\frac{12}{16} + \frac{4}{16} = \frac{32}{16} = 2 \\ E\left(X^2\right) &= \sum_x x^2 p(x) = 0^2\binom{4}{0}\frac{1}{16} + 1^2\binom{4}{1}\frac{1}{16} + 2^2\binom{4}{2}\frac{1}{16} + 3^2\binom{4}{3}\frac{1}{16} + 4^2\binom{4}{4}\frac{1}{16} \\ &= \frac{0}{16} + \frac{4}{16} + \frac{24}{16} + \frac{36}{16} + \frac{16}{16} = \frac{80}{16} = 5 \\ V(X) &= E\left(X^2\right) - [E(X)] = 5 - 2^2 = 1 \end{aligned}$$

b. Using the fact that X has a binomial distribution with $n = 4$ and $p = \frac{1}{2}$:

$$E(X) = np = 4 \cdot \frac{1}{2} = 2 \qquad V(X) = np(1-p) = 4 \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right) = 1 \qquad \square$$

c. The average value of the three odd faces is $\frac{1+3+5}{3} = \frac{9}{3} = 3$. From the answer to **b**, we woud expect two odd faces to turn up in four rolls, ans so their sum should be expected to be $3 + 3 = 6$, *i.e.* $E(Y) = 6$. ■

2. A continuous random variable Z is uniformly distributed between 0 and 3.

a. What is the expected value of Z^2? *[10]*

b. What is the probability that Z^2 is at most 8? *[5]*

SOLUTIONS. **a.** The probability density function of a random variable Z that is uniformly distributed between 0 and 3 is $f(z) = \begin{cases} \frac{1}{3} & 0 \le z \le 3 \\ 0 & \text{otherwise} \end{cases}$. Then

$$\begin{aligned} E\left(Z^2\right) &= \int_{-\infty}^{\infty} z^2 f(z)\,dz = \int_{-\infty}^{0} z^2 \cdot 0\,dz + \int_{0}^{3} z^2 \cdot \frac{1}{3}\,dz + \int_{3}^{\infty} z^2 \cdot 0\,dz \\ &= 0 + \frac{1}{3} \cdot \left.\frac{z^3}{3}\right|_0^3 + 0 = \frac{3^3}{9} - \frac{0^3}{9} = \frac{27}{9} - 0 = 3\,. \qquad \square \end{aligned}$$

b. Note that since $Z \ge 0$, $Z^2 \le 8 \Leftrightarrow Z \le \sqrt{8} = 2\sqrt{2}$. It follows that

$$\begin{aligned} P\left(Z^2 \le 8\right) = P\left(2\sqrt{2}\right) &= \int_{-\infty}^{2\sqrt{2}} f(z)\,dz = \int_{-\infty}^{0} 0\,dz + \int_{0}^{2\sqrt{2}} \frac{1}{3}\,dz \\ &= 0 + \frac{1}{3} \cdot z\Big|_0^{2\sqrt{2}} = \frac{2\sqrt{2}}{3} - \frac{0}{3} = \frac{2\sqrt{2}}{3}\,. \qquad \blacksquare \end{aligned}$$

3. A hand of five cards is drawn simultaneously (*i.e.* without order or replacement) from a standard 52-card deck.

a. What is the probability that the hand is a "full house," that is, that it includes a pair of one kind and a triple of another kind? *[8]*

b. What is the probability that all the cards in the hand are of different kinds? *[7]*

SOLUTIONS. **a.** There are $\binom{52}{5} = 2598960$ hands of five cards that can be drawn from a standard deck without order or replacement. There are $\binom{13}{1} = 13$ ways to choose the kind for the pair and $\binom{4}{2} = 6$ ways to choose two of the four cards of that kind. Similarly, there are $\binom{12}{1} = 12$ ways to choose another kind for the triple and $\binom{4}{3} = 4$ ways to choose a three of the four cards of that kind. It follows that there are $13 \cdot 6 \cdot 12 \cdot 4 = 3744$ "full house" hands, and so the probability that a given hand is a "full house" is $\dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}}{\binom{52}{5}} = \dfrac{3744}{2598960} \approx 0.00144$. □

b. As noted above in the soltion to part **a**, there are $\binom{52}{5} = 2598960$ possible five -card hands. To count the number of these that have cards all of different kinds, note that there are $\binom{13}{5} = 1287$ ways to choose five of thirteen kinds, and then $\binom{4}{1} = 4$ ways to choose one of four cards for each kind, for a total of $\binom{13}{5} \cdot 4^5 = 1287 \cdot 1024 = 1317888$. Thus the probability that a random five-card hand will consist of cards which are all of different kinds is $\dfrac{\binom{13}{1} \cdot 4^5}{\binom{52}{5}} = \dfrac{1317888}{2598960} \approx 0.50708$. ■

4. Suppose Y is a normally distributed continuous random variable with expected value $\mu = 3$ and standard deviation $\sigma = 2$.

a. Compute $P(Y \geq 0)$ using your standard normal table. *[5]*

b. Find the median of Y, *i.e.* the number m such that $P(Y \leq m) = 0.5$, with the help of your standard normal table. *[5]*

Solutions. **a.** Recall that Y has a normal distribution with $\mu = 3$ and $\sigma = 2$ if and only if $Z = \frac{Y-\mu}{\sigma} = \frac{Y-3}{2}$ has a standard normal distribution. Since $Y \geq 0$ exactly when $Z = \frac{Y-3}{2} \geq \frac{0-3}{2} = -\frac{3}{2} = -1.5$, we have $P(Y \geq 0) = P(Z \geq -1.5)$. By the symmetry of the standard normal distribution and our table for it, we then have $P(Y \geq 0) = P(Z \geq -1.5) = P(Z \leq 1.5) = 0.5 + P(0 \leq Z \leq 1.5) \approx 0.5 + 0.4332 = 0.9332$. □

b. As noted in the slution to part **a**, Y has a normal distribution with $\mu = 3$ and $\sigma = 2$ if and only if $Z = \frac{Y-\mu}{\sigma} = \frac{Y-3}{2}$ has a standard normal distribution. Since $Y \leq m$ exactly when $Z = \frac{Y-3}{2} \leq \frac{m-3}{2}$, we need to find the m such that $P\left(Z \leq \frac{m-3}{2}\right) = 0.5$. For the standard normal distribution $P(Z \leq 0) = 0.5$, so $\frac{m-3}{2} = 0 \Rightarrow m - 3 = 0 \Rightarrow m = 3$. ■

5. A fair coin is tossed, and then tossed some more until it comes up again with whatever face came up on the first toss.

a. What are the sample space and probability function? *[7]*

b. Let A be the event that no more than three tosses took place and let B be the event that the second toss was a head. Determine whether the events A and B are independent or not. *[8]*

Solutions. **a.** The sample space S consists of all the sequences of heads and tails that are at least two long and such that the first and last tosses are the same and different from all the tosses in between:

$$S = \{ HH, TT, HTH, THT, HTTH, THHT, HTTTH, THHHT, \dots \}$$

Since the coin is fair, each sequence of heads and/or tails of length n has probability equal to $\frac{1}{2} \cdot \frac{1}{2} \cdot \dots \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n$. Thus the probability function $p : S \to \mathbb{R}$ is: $p(HH) = p(TT) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$, $p(HTH) = p(THT = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, $p(HTTH) = p(THHT) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$, … In general, for $n \geq 2$, $p\left(HT^{n-2}H\right) = p\left(TH^{n-2}T\right) = \left(\frac{1}{2}\right)^n$. □

b. Since there must be a second toss, and since its outcome must be independent of any other because the coin is fair, $P(B) = \frac{1}{2} = 0.5$. Since $A = \{ HH, TT, HTH, THT \}$, $P(A) = p(HH) + p(TT) + p(HTH) + p(THT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{3}{4} = 0.75$, and since $A \cap B = \{ HH, THT \}$, $P(A \cap B) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375$. Thus $P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$, so A and B are independent events. ■

[Parts ♢ and ♣ are on page 2.]

Part $\diamondsuit$. Do any *two* (2) of **6–10**. *[Subtotal = 30/100]*

6. Suppose Z is a continuous random variable with a standard normal distribution.

a. Compute $P(-2 < Z < 2)$ using your standard normal table. *[5]*

b. Use Tschebysheff's Inequality to show that $P(-2 \leq Z < 2) \geq \frac{3}{4}$. *[5]*

c. Without computing the relevant integral, explain why $\mathrm{E}(Z) = 0$. *[5]*

SOLUTIONS. **a.** Since Z has a standard normal distribution,

$$\begin{aligned}
P(-2 < Z < 2) &= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-2}^{0} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-z^2/2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-z^2/2}\, dz \quad \begin{array}{l}(\text{As } e^{-(-z)^2/2} = e^{-z^2/2} \\ \text{for all } z.)\end{array} \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-z^2/2}\, dz \approx 2 \cdot 0.4772 = 0.9544\,,
\end{aligned}$$

using the table to get $\displaystyle\frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-z^2/2}\, dz \approx 0.4772$. $\square$

b. Recall that Tschebysheff's Inequality tells us that if the random variable X has expected value μ and (finite!) variance σ^2, then, for any $\epsilon > 0$, $P\left(|X - \mu| \geq \epsilon\right) \leq \dfrac{\sigma^2}{\epsilon^2}$. In the case of the normal standard distribution, we have $\mu = 1$ and $\sigma^2 = 1$, and we will take $\epsilon = 2$. Since $P(-2 \leq Z < 2) = 1 - P\left(|Z - 0| \geq 2\right)$ and, by Tschebysheff's Inequality, $P\left(|Z - 0| \geq 2\right) \leq \frac{1}{2^2} = \frac{1}{4}$, it follows that $P(-2 \leq Z < 2) \geq 1 - \frac{1}{4} = \frac{3}{4}$, as desired. $\square$

c. The probability density function $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ of the standard normal distribution is even, *i.e.* $e^{-(-z)^2/2} = e^{-z^2/2}$ for all z (as noted in the solution to part **a**), which means that there is just as much area under the graph to the left of 0 as there is to the right of 0. It follows that $E(Y) = 0$. $\blacksquare$

7. A fair standard six-sided die is rolled twice.

a. What is the probability of a sum of 6 for the two rolls, given that the two rolls gave different numbers? *[7]*

b. Determine whether the events that the two rolls are different and that the sum of the two rolls is 7 are independent or not. *[8]*

SOLUTIONS. **a.** First, note that the sample space is $S\,\{\,(1,1),\ (1,2),\ (2,1),\ (1,3),\ (2,3),\ \ldots,(5,6),\ (6,6)\,\}$, which consists of $6 \cdot 6 = 36$ equally likely outcomes.

Let A be the event that the two rolls have a sum of 6. Thus $A = \{\,(1,5),\ (2,4),\ (3,3),\ (4,2),\ (5,1)\,\}$, so it consists of 5 of the outcomes in S. Let B be the event that the two rolls gave different numbers. Then $B = S \setminus \{\,(1,1),\ (2,2),\ \ldots,(6,6)\,\}$, so it consists of $36 - 6 = 30$ of the outcomes in S. Finally, $A \cap B = \{\,(1,5),\ (2,4),\ (4,2),\ (5,1)\,\}$, which includes 4 outcomes. By definition, the probability of a sum of 6 for the two rolls, given that the two rolls gave different numbers, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{4/36}{30/36} = \frac{4}{30} = \frac{2}{15} = 0.133\dot{3}\,.$$

Note that because the outcomes in our finite sample space S are equally likely, we can compute the probability of an event by dividing the number of outcomes in the event by the number of outcomes in the sample space. □

b. Let B be the event that the two rolls are different, as in the solution to part **a** above, and let C be the event that the sum of the two rolls is 7. Then $C = \{\,(1,6),\ (2,5),\ (3,4),\ (4,3),\ (5,2)\ (6,1)\,\}$ includes 6 of the outcomes in the sample space, every one of which has the two rolls different. It follows that $B \cap C = C$. Since

$$P(B \cap C) = P(C) = \frac{6}{36} \neq \frac{5}{36} = \frac{30}{36} \cdot \frac{6}{36} = P(B)P(C)\,,$$

the events B and C are not independent. □

8. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	0	1	2
1	0.2	0.1	0.1
2	0.1	0	0.1
3	0.1	0.1	0.2

a. Compute $E(X)$, $E(Y)$, $V(X)$, $V(Y)$, and $\mathrm{cov}(X,Y)$. *[10]*

b. Let $U = X - 2Y$. Compute $E(U)$ and $V(U)$. *[5]*

SOLUTIONS. **a.** Here goes:

$$\begin{aligned}
E(X) &= 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) \\
&= 1 \cdot (0.2 + 0.1 + 0.1) + 2 \cdot (0.1 + 0 + 0.1) + 3 \cdot (0.1 + 0.1 + 0.2) \\
&= 1 \cdot 0.4 + 2 \cdot 0.2 + 3 \cdot 0.4 = 0.4 + 0.4 + 1.2 = 2 \\
E(Y) &= 0 \cdot P(Y=0) + 1 \cdot P(Y=1) + 2 \cdot P(Y=2) \\
&= 0 \cdot (0.2 + 0.1 + 0.1) + 1 \cdot (0.1 + 0 + 0.1) + 2 \cdot (0.1 + 0.1 + 0.2) \\
&= 0 \cdot 0.4 + 1 \cdot 0.2 + 2 \cdot 0.4 = 0 + 0.2 + 0.8 = 1 \\
E\left(X^2\right) &= 1^2 \cdot P(X=1) + 2^2 \cdot P(X=2) + 3^2 \cdot P(X=3) \\
&= 1 \cdot 0.4 + 4 \cdot 0.2 + 9 \cdot 0.4 = 0.4 + 0.8 + 3.6 = 4.8 \\
E\left(Y^2\right) &= 0^2 \cdot P(Y=0) + 1^2 \cdot P(Y=1) + 2^2 \cdot P(Y=2) \\
&= 0 \cdot 0.4 + 1 \cdot 0.2 + 4 \cdot 0.4 = 0 + 0.2 + 1.6 = 1.8 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 4.8 - 2^2 = 4.8 - 4 = 0.8 \\
V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 1.8 - 1^2 = 1.8 - 1 = 0.8 \\
E(XY) &= 1 \cdot 0 \cdot P(X=1\ \&\ Y=0) + 1 \cdot 1 \cdot P(X=1\ \&\ Y=1) + \\
&\qquad \cdots + 3 \cdot 2 \cdot P(X=3\ \&\ Y=2) \\
&= 0 \cdot 0.2 + 1 \cdot 0.1 + 2 \cdot 0.1 + 0 \cdot 0.1 + 2 \cdot 0 + 4 \cdot 0.1 + 0 \cdot 0.1 + 3 \cdot 0.1 + 6 \cdot 0.2 \\
&= 0 + 0.1 + 0.2 + 0 + 0 + 0.4 + 0 + 0.3 + 1.2 = 2.2 \\
\mathrm{cov}(X,Y) &= E(XY) - E(X)E(Y) = 2.2 - 2 \cdot 1 = 0.2 \quad \square
\end{aligned}$$

b. The expected value operator is linear, so, exploiting our work in the solution to part **a**,

$$E(U) = E(X - 2Y) = E(X) - 2E(Y) = 2 - 2 \cdot 1 = 2 - 2 = 0\,.$$

To compute $V(U)$, we first compute $E\left(U^2\right)$:

$$\begin{aligned} E\left(U^2\right) &= E\left([x - 2y]^2\right) E\left(X^2 - 2XY + 4Y^2\right) = E\left(X^2\right) - 2E(XY) + 4E\left(Y^2\right) \\ &= 4.8 - 2 \cdot 2.2 + 4 \cdot 1.8 = 4.8 - 4.4 + 7.2 = 7.6 \end{aligned}$$

Thus

$$V(U) = E\left(U^2\right) - [E(U)]^2 = 7.6 - 0^2 = 7.6 - 0 = 7.6\,. \qquad \square$$

9. Suppose A, B, and C are events such that $P(ABC) = P(A)P(B)P(C)$. Does it follow that the first two events, A and B, are independent? Show that it does or give an example to show that it doesn't. *[15]*

Solution. A and B do not have to be independent. Suppose, for example, that we have twenty balls, numbered 1 through 20, from which we choose one at random. The sample space is thus $S = \{1, 2, \ldots, 20\}$ and the probability function is $p(n) = \frac{1}{20} = 0.05$ for $1 \leq n \leq 20$. Let $A = \{1, 2, \ldots, 10\}$, so $P(A) = \frac{10}{20} = \frac{1}{2} = 0.5$, and let $B = \{9, 10, \ldots 18\}$, so $P(B) = \frac{10}{20} = \frac{1}{2} = 0.5$, too. $A \cap B = \{9, 10\}$, so $P(A \cap B) = \frac{2}{20} = \frac{1}{10} = 0.1$, but $P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25 \neq 0.1 = \frac{1}{10}$, so A and B are not independent. Now let $C = \{6, 7, 8, 9\}$, so $P(C) = \frac{4}{20} = \frac{1}{5}$. As $A \cap B \cap C = \{9\}$, we have $P(A \cap B \cap C) = \frac{1}{20} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{5} = P(A)P(B)P(C)$. ■

Note. **9** is probably the hardest question on the exam ...

10. Let $f(y) = \begin{cases} \frac{1}{3}y^2 & -1 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$ be the probability density function of the continuous random variable Y.

a. Verify that $f(y)$ is indeed a probability density function. *[7]*

b. Compute $E(Y)$ and $V(Y)$. *[8]*

Solutions. **a.** Since $\frac{1}{3}y^2 \geq 0$ for any y and $0 \geq 0$, it is pretty obvious that $f(y) \geq 0$ for all y. Since we also have that

$$\begin{aligned} \int_{-\infty}^{\infty} f(y)\,dy &= \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{2} \frac{1}{3}y^2\,dy + \int_{2}^{\infty} 0\,dy = 0 + \frac{1}{3} \cdot \left.\frac{y^3}{3}\right|_{-1}^{2} + 0 \\ &= \left.\frac{y^3}{9}\right|_{-1}^{2} = \frac{2^3}{9} - \frac{(-1)^3}{9} = \frac{8}{9} - \frac{-1}{9} = \frac{8}{9} + \frac{1}{9} = 1\,, \end{aligned}$$

it follows $f(y)$ is a probability density function by definition. $\square$

b. By definition:

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} y f(y)\,dy = \int_{-\infty}^{-1} y \cdot 0\,dy + \int_{-1}^{2} y \cdot \frac{1}{3}y^2\,dy + \int_{2}^{\infty} y \cdot 0\,dy \\
&= \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{2} \frac{1}{3}y^3\,dy + \int_{2}^{\infty} 0\,dy = 0 + \frac{1}{3} \cdot \left.\frac{y^4}{4}\right|_{-1}^{2} + 0 = \left.\frac{y^4}{12}\right|_{-1}^{2} \\
&= \frac{2^4}{12} - \frac{(-1)^4}{12} = \frac{16}{12} - \frac{1}{12} = \frac{15}{12} = \frac{5}{4} = 1.25
\end{aligned}$$

To compute the vaiance of Y, we will use the formula $V(Y) = E\left(Y^2\right) - [E(Y)]^2$. We just computed $E(Y)$, but we still need to compute $E\left(Y^2\right)$ to use this formula. By definition:

$$\begin{aligned}
E\left(Y^2\right) &= \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_{-\infty}^{-1} y^2 \cdot 0\,dy + \int_{-1}^{2} y^2 \cdot \frac{1}{3}y^2\,dy + \int_{2}^{\infty} y^2 \cdot 0\,dy \\
&= \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{2} \frac{1}{3}y^4\,dy + \int_{2}^{\infty} 0\,dy = 0 + \frac{1}{3} \cdot \left.\frac{y^5}{5}\right|_{-1}^{2} + 0 = \left.\frac{y^5}{15}\right|_{-1}^{2} \\
&= \frac{2^5}{15} - \frac{(-1)^5}{15} = \frac{32}{15} - \frac{-1}{15} = \frac{32}{15} + \frac{1}{15} = \frac{33}{15} = \frac{11}{5} = 2.2
\end{aligned}$$

It follows that

$$V(Y) = E\left(Y^2\right) - [E(Y)]^2 = \frac{11}{5} - \left(\frac{5}{4}\right)^2 = \frac{11}{5} - \frac{25}{16} = \frac{176}{80} - \frac{125}{80} = \frac{51}{80} = 0.6375\,.$$

Whew! ■

[Total = 100]

Part ♣. Bonus!

K. If two fair standard six-sided dice are rolled together three times, what is the probability that the two dice will match (*i.e.* have the same number come up) on one of the three rolls? *[1.5]*

Solution. If the two dice are rolled once there are $6 \cdot 6 = 36$ equally likely outcomes, of which 6 are matches, so the probability that the two dice will match on a single roll is $\frac{6}{36} = \frac{1}{6}$. The probability that they will match exactly once in three rolls is therefore $\binom{3}{1}\frac{1}{6}\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6}\right) = \frac{75}{216} = \frac{25}{72} \approx 0.3472$. On the other hand, the probability that they will match at least once [nice ambiguity in the question!] is 1 minus the probability that they never match, *i.e.* $1 - \left(1-\frac{1}{6}\right)\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6}\right) = \frac{91}{216} \approx 0.4123$. ■

Q. Write an original little poem about probability or mathematics in general. *[1.5]*

Solution. You're on your own on this one! ■

I hope this went better than the test did. :-)

Enjoy the rest of the summer!