Trent University

MATH 1550H Test

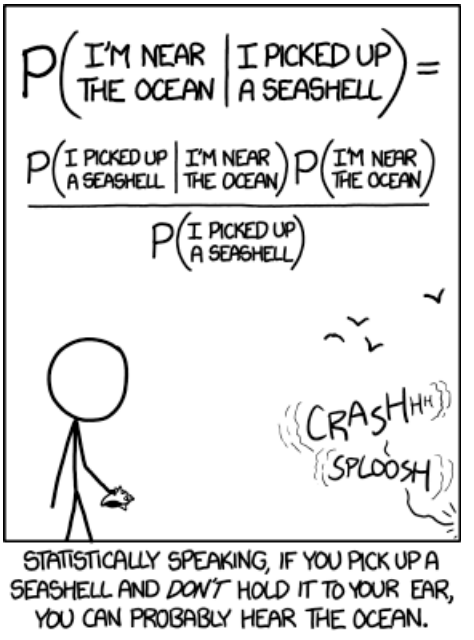
15 July, 2013. *Time: 60 minutes*

Name: Solutions

Student Number: 0123456

Bonus. Is the equation given in the comic at right correct? Explain why or why not. *[1]*

[Taken from the webcomic *xkcd* on 2013.07.10. It can be found at `http://xkcd.com/1236/` and is licensed under a Creative Commons Attribution-NonCommercial 2.5 License.]



Question	Mark
1	___
2	___
3	___
Bonus	___
Total	___

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.
- You need not simplify numerical answers unless it's easy to do ...

Solution to the **Bonus**. It's correct. If A is the event "I'm near the ocean" and B is the event "I picked up a seashell", then we need to verify $P(A \mid B) = \frac{P(B|A)P(A)}{P(B)}$:

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{\frac{P(AB)}{P(A)}P(A)}{P(B)} = \frac{\frac{P(BA)}{P(A)}P(A)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B)} \qquad \blacksquare$$

1. Do any *three (3)* of **a**–**d**. *[12 = 3 × 4 each]*

a. A number is chosen at random from $\{1, 2, 3, \dots, 100\}$. Let A be the event that the number is divisible by 5 and let B be the event that the number is divisible by 3. Determine whether A and B are independent or not.

b. A fair coin is tossed five times. What is the probability that exactly two heads will occur in the five tosses?

c. Three cards are drawn at random, one at a time and without replacement, from a standard 52-card deck. What is the probability that all three are from different suites?

d. How many different ways are there to rearrange the letters in the word "unusual" if there is no way to tell the "u"s apart?

Solutions. **a.** A consists of the outcomes 5, 10, 15, ..., 100, of which there are 20, B consists of the outcomes 3, 6, 9, ..., 99, of which there are 33, and AB consists of the outcomes 15, 30, 45, ..., 90, of which there are 6. Since any outcome is as likely as any other and there are 100 outcomes in the sample space, it follows that $P(A) = \frac{20}{100} = \frac{1}{5}$, $P(B) = \frac{33}{100}$, and $P(AB) = \frac{6}{100} = \frac{3}{50}$. It follows that

$$P(AB) = \frac{3}{50} = \frac{30}{500} \neq \frac{33}{500} = \frac{1}{5} \cdot \frac{33}{100} = P(A)P(B)\,,$$

so A and B are *not* independent. □

b. There are $2^5 = 32$ total outcomes in the sample space, and there are $\binom{5}{2} = 10$ ways to pick two positions for the heads (and thus that many outcomes with exactly two heads). Hence $P(\text{exactly two heads in five tosses}) = \binom{5}{2}/2^5 = \frac{10}{32} = \frac{5}{16} = 0.3125$. □

c. There are $52 \cdot 51 \cdot 50$ total outcomes, since order matters here. There are $52 \cdot 39 \cdot 26$ outcomes in which the three cards are of different suites: you have 52 choices for the first card, $39 = 52 - 13$ choices for the second, since it must be of a different suite, and $26 = 52 - 2 \cdot 13$ for the third, since it must be of yet another suite. It follows that $P(\text{all three cards are of a different suite}) = \frac{52 \cdot 39 \cdot 26}{52 \cdot 51 \cdot 50} = \frac{13 \cdot 13}{17 \cdot 25} = \frac{169}{425} \approx 0.397647$. □

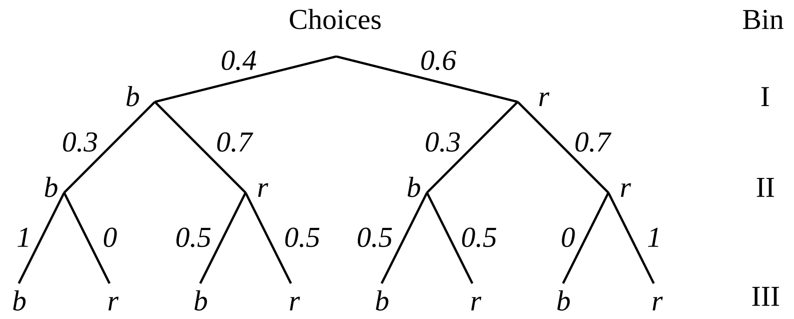
d. There 7 letters (counting all three "u"s) in the word "unusual". In rearranging the letters, there are $\binom{7}{3}$ ways to pick the three positions to be occupied by the indistinguishable "u"s, and $4! = 4 \cdot 3 \cdot 2 \cdot 1$ ways to fill the other four positions with the four letters. This gives a total of $\binom{7}{3} \cdot 4! = \frac{7!}{3!4!} \cdot 4! = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ ways to rearrange the letters if the "u"s cannot be distinguished. ■

2. Do any *one (1)* of **a** or **b**. *[8 = 1 × 8 each]*

a. Initially, Bin I has four blue and six red balls, Bin II has three blue and seven red balls, and Bin III is empty. A ball is chosen at random from each of Bins I and II and put into Bin III. One of the two balls now in Bin III is then chosen at random. What is the probability that the ball chosen from Bin III is blue? What is the probability that the ball chosen from Bin III is blue, given that the ball chosen from Bin I is red?

b. A fair coin is tossed four times. Let X be the number of heads that occur in the four tosses. Compute the expected value, $E(X)$, and standard deviation, σ_X, of X.

SOLUTIONS. **a.** Here is a tree diagram of the process:



Following the branches that lead to getting a blue ball from Bin III and taking their probabilities, we see that

$$\begin{aligned} P(\text{blue from Bin III}) &= 0.4 \cdot 0.3 \cdot 1 + 0.4 \cdot 0.7 \cdot 0.5 + 0.6 \cdot 0.3 \cdot 0.5 + 0.6 \cdot 0.7 \cdot 0 \\ &= 0.12 + 0.14 + 0.09 + 0 = 0.35\,. \end{aligned}$$

Following the branches that lead to getting a red ball from Bin I and a blue ball from Bin III, and applying the definition of conditional probability, we see that

$$\begin{aligned} P\left(\text{blue from Bin III} \mid \text{red from Bin I}\right) &= \frac{P\left(\text{blue from Bin III \& red from Bin I}\right)}{P\left(\text{red from Bin I}\right)} \\ &= \frac{0.6 \cdot 0.3 \cdot 0.5 + 0.6 \cdot 0.7 \cdot 0}{0.6} = 0.15\,. \qquad \Box \end{aligned}$$

b. Note that there are $2^4 = 16$ equally likely outcomes in the underlying sample space. We'll compute the values of the probability mass function by counting how many outcomes have a given number of heads, which we'll be do by counting how many ways there are to pick the positions in the four tosses that the heads show up in. Thus $\binom{4}{k}$ outcomes have exactly k heads. It follows that:

x	0	1	2	3	4	otherwise
$p(x)$	$\frac{\binom{4}{0}}{2^4} = \frac{1}{16}$	$\frac{\binom{4}{1}}{2^4} = \frac{4}{16}$	$\frac{\binom{4}{2}}{2^4} = \frac{6}{16}$	$\frac{\binom{4}{3}}{2^4} = \frac{4}{16}$	$\frac{\binom{4}{4}}{2^4} = \frac{1}{16}$	0

It follows that $E(X) = \sum_{x=0}^{4} xp(x) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = \frac{0+4+12+12+4}{16} = \frac{32}{16} = 2.$

To compute $\sigma_X = \sqrt{\text{Var}_X}$ we need $\text{Var}_X = E\left((X - E(X))^2\right) = E\left(X^2\right) - (E(X))^2$. Since $E\left(X^2\right) = \sum_{x=0}^{4} x^2 p(x) = 0^2 \cdot \frac{1}{16} + 1^2 \cdot \frac{4}{16} + 2^2 \cdot \frac{6}{16} + 3^2 \cdot \frac{4}{16} + 4^2 \cdot \frac{1}{16} = \frac{0+4+24+36+16}{16} = \frac{80}{16} = 5$, $\text{Var}_X = E\left(X^2\right) - (E(X))^2 = 5 - 2^2 = 1$, and thus $\sigma_X = \sqrt{\text{Var}_X} = \sqrt{1} = 1$. ■

3. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Show that if events A and B are independent, then A and B^c are independent too.

b. A bin contains four green and six purple balls. What is the expected number of green balls if three balls are chosen randomly, with replacement, from the bin?

c. A fair die is rolled twice. What is the probability that both rolls came up 5, given that the sum of the two rolls is a number divisible by 5?

Solutions. **a.** Recall that A and B are independent exactly when $P(AB) = P(A)P(B)$. Then, since $A - B = A - AB$ (think about it!) and $AB \subseteq A$, we have

$$\begin{aligned} P(A)P\left(B^c\right) &= P(A)\left(1 - P(B)\right) = P(A) - P(A)P(B) = P(A) - P(AB) \\ &= P\left(A - AB\right) = P(A - B) = P\left(AB^c\right), \end{aligned}$$

so A and B^c are also independent. □

b. Let X be the number of green balls chosen in the process. Since we are choosing with replacement, each ball has a probability of $\frac{4}{10} = 0.4$ of being green and a probability of $\frac{6}{10} = 0.6$ of being purple. For each x in $\{0, 1, 2, 3\}$, $p(x) = P(X = x)$ will equal the number of ways of picking $\binom{3}{x}$ positions for the x green balls, times the probability of filling each of the x positions with a green ball, $\left(\frac{4}{10}\right)^x$, times the probability of filling the remaining $3 - x$ positions with a purple ball, $\left(\frac{6}{10}\right)^{3-x}$. It follows that

$$\begin{aligned} E(X) = \sum_{x=0}^{3} xp(x) &= 0 \cdot \binom{3}{0}\left(\frac{4}{10}\right)^0\left(\frac{6}{10}\right)^3 + 1 \cdot \binom{3}{1}\left(\frac{4}{10}\right)^1\left(\frac{6}{10}\right)^2 \\ &\quad + 2 \cdot \binom{3}{2}\left(\frac{4}{10}\right)^2\left(\frac{6}{10}\right)^1 + 3 \cdot \binom{3}{3}\left(\frac{4}{10}\right)^3\left(\frac{6}{10}\right)^0 \\ &= \frac{0 \cdot 1 \cdot 1 \cdot 216 + 1 \cdot 3 \cdot 4 \cdot 36 + 2 \cdot 3 \cdot 16 \cdot 6 + 3 \cdot 1 \cdot 64 \cdot 1}{1000} \\ &= \frac{1200}{1000} = 1.2 \qquad \square \end{aligned}$$

c. We'll do this one with the "cutting down the sample space" trick. The possible sums divisible by 5 are 5 and 10, so the seven – equally likely! – outcomes with a sum divisible by 5 are $(1,4)$, $(4,1)$, $(2,3)$, $(3,2)$, $(4,6)$, $(6,4)$, and $(5,5)$. It follows that

$$P\left(\,(5,5) \mid \text{sum divisible by } 5\,\right) = \frac{1}{7}. \qquad \blacksquare$$

[Total = 30]