Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2013

Assignment #3
Fairness from bias

1. You are given a biased coin which comes up heads $\frac{2}{3}$ of the time and tails $\frac{1}{3}$ of the time. What you would like to have is a fair coin. How can you use the biased coin to deliver an event which has a probability of $\frac{1}{2}$? (You may toss the biased coin as many times as necessary for whatever process you devise.) *[10]*

SOLUTION. Here's the basic process (apparently due to John von Neumann) for simulating a fair coin with a biased coin:

1. Toss the biased coin twice.
2. If the result of step 1 is
   HT, then return "head";
   TH, then return "tail";
   otherwise, repeat step 1.

Since $P(\text{head}) = P(HT) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} = \frac{1}{3} \cdot \frac{2}{3} = P(TH) = P(\text{tail})$, $P(\text{head} \mid \text{head or tail}) = P(\text{tail} \mid \text{head or tail}) = \frac{1}{2}$. (*e,g.* $P(\text{head} \mid \text{head or tail}) = \frac{P(\text{head})}{P(\text{head or tail})} = \frac{P(\text{head})}{P(\text{head})+P(\text{tail})} = \frac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{9}} = \frac{1}{2}$)

There is one potential glitch here: it may take a few tries before you get HT or TH in a pair of tosses, especially if the coin is greatly biased. In principle, you could be tossing forever, though that has probability 0:

$$P(n \text{ double tosses with HH or TT each time}) = \left(\frac{5}{9}\right)^n ,$$

which has a limit of 0 as $n \to \infty$ since $\left|\frac{5}{9}\right| < 1$. ■