Mathematics 1550H – Introduction to probability
Trent University, Summer 2013

Solutions to the Quizzes

Quiz #1. Wednesday, 26 June, 2013. *[15 minutes]*

A fair (standard six-sided) die is tossed twice. Let b and c be the outcomes of the first and second tosses respectively. Since the die is fair, any pair (b, c) is as likely to turn up as any other.

1. What is the sample space S, exactly? How many possible outcomes are there in S? *[1]*
2. If E is the event that $b = 1$, determine $P(E)$. *[2]*
3. If F is the event that $b^2 \geq 4c$, determine $P(E)$. *[3]*

Solution to 1. The sample space consists of all ordered pairs (b, c), where b and c are both integers between 1 and 6 inclusive. For lovers of notation:

$$S = \{\,(b, c) \mid b,\ c \in \mathbb{N} \ \&\ 1 \leq b \leq 6 \ \&\ 1 \leq c \leq 6\,\}$$

Since there are 6 choices for each of b and c, and b and c are independently determined, there are $6 \cdot 6 = 36$ possible outcomes $(b, c) \in S$. ■

Solution to 2. If E is the event that $b = 1$, then E consists of all the outcomes (b, c) where $b = 1$, *i.e.* $E = \{\,(1, 1),\ (1, 2),\ (1, 3),\ (1, 4),\ (1, 5),\ (1, 6)\,\}$, of which there are six. Since there are 36 outcomes in S, which are all equally likely, each individual outcome has $\frac{1}{36}$ chance of occurring. It follows that $P(E) = 6 \cdot \frac{1}{36} = \frac{1}{6}$. ■

Solution to 3. As in the solution to question 2 above, our strategy is to figure out how many of the possible outcomes are in F:

If $b = 1$, then $b^2 = 1 < 4 \leq 4c$ for all $1 \leq c \leq 6$, so no outcomes with $b = 1$ make it into F.
If $b = 2$, then $b^2 = 4 = 4 \cdot 1$, but $b^2 = 4 < 4c$ for all $2 \leq c \leq 6$, so only one outcome with $b = 2$ makes it into F.
If $b = 3$, then $b^2 = 9 \geq 4c$ for $c = 1$ and 2, but $b^2 = 4 < 4c$ for all $3 \leq c \leq 6$, so only two outcomes with $b = 3$ make it into F.
If $b = 4$, then $b^2 = 16 \geq 4c$ for $1 \leq c \leq 4$, but $b^2 = 4 < 4c$ for all $5 \leq c \leq 6$, so four outcomes with $b = 3$ make it into F.
If $b = 5$ or 6, then $b^2 \geq 25 \geq 24 = 4 \cdot 6 \geq 4c$ for all $1 \leq c \leq 6$, so $6 + 6 = 12$ outcomes with $b = 5$ or 6 make it into F.

It follows that F consists of $0 + 1 + 2 + 4 + 6 + 6 = 19$ outcomes, and so $P(F) = 19 \cdot \frac{1}{36} = \frac{19}{36}$. ■

Quiz #2. Wednesday, 3 July, 2013. *[15 minutes]*

A fair two-sided coin has a thick flat edge. If it is tossed, it can land heads up (H) or tails up (T) with equal likelihood, or land on edge (E), which occurs with a probability of $0.01 = \frac{1}{100}$. With repeated tosses, their *sum* is computed by counting each H as 1 and each E as $\frac{1}{2}$ (*i.e.* $H = 1$, $E = \frac{1}{2}$, and $T = 0$). The coin is tossed three times. Compute the probability of the following events:

1. The coin landed on edge twice in the three tosses. *[2]*
2. The sum of the three tosses is an even integer. *[4]*

SOLUTION TO 1. Any particular sequence of two E's and one H or T has a probability of $0.01 \times 0.01 \times \frac{1-0.01}{2} = 0.0001 \times 0.495 = 0.0000495$ of occurring. There are 3 ways of picking the non-E position in the sequence and 2 ways to fill that position, so there are $3 \times 2 = 6$ such sequences. It follows that

$$P\left(\text{The coin landed on edge twice in the three tosses.}\right) = 6 \times 0.0000495 = 0.000297\,. \qquad \blacksquare$$

NOTE: Them's not good betting odds, partner ...

SOLUTION TO 2. Note first that the possible sums of three tosses are 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and 3. The even integers on this list are 0 and 2.

There is only one way to get a sum of 0, toss 3 Ts in a row, which has a probability of $0.495 \times 0.495 \times 0.495 = 0.495^3 = 0.121287375$.

To get a sum of 2, one can either toss 2 Hs and a T in some order, or toss 1 H and 2 Es in some order. In the former case, there are 3 ways to pick the spot in the sequence in which the T occurs, and any such sequence has a probability of $0.495 \times 0.495 \times 0.495 = 0.495^3 = 0.121287375$ of occurring. In the latter case, there are 3 ways to pick the spot in the sequence in which the H occurs, and any such sequence has a probability of $0.495 \times 0.01 \times 0.01 = 0.0000495$ of occurring. Thus the probability of obtaining a sum of 2 is:

$$3 \times 0.121287375 + 3 \times 0.0000495 = 0.363862125 + 0.0001485 = 0.364010625$$

It follows that

$$\begin{aligned} P\left(\text{The sum of the three tosses is an even integer.}\right) &= 0.121287375 + 0.364010625 \\ &= 0.485298\,. \qquad \blacksquare \end{aligned}$$

NOTE: Not quite even odds.

NOTE: *It's OK with me to leave the answers unsimplified!*

Quiz #3. Monday, 8 July, 2013. *[15 minutes]*

You are pet-sitting three dogs and two cats. Two of the dogs and one cat are black, and the remaining dog and cat are white. When you let them out into the backyard, they go out one at a time in random order.

1. What is the probability that first two pets to go out were both black? *[2]*
2. If one of the the last two pets to go out was white and the other was black, what is the probability that first two to go out were both black? *[4]*

SOLUTION TO 1. There are a total of $5! = 120$ possible orders in which the five pets could exit, all equally likely. There are $\binom{3}{2} = 3$ ways to pick two of the three black pets, and $2! = 2$ ways to arrange the two in some order. The remaining three pets can be arranged in $3! = 6$ different ways. It follows that there are $\binom{3}{2} \cdot 2! \cdot 3! = 3 \cdot 2 \cdot 6 = 36$ orders in which the pets can exit in which the first two are black. Thus

$$P\,(\text{The first two pets are black.}) = \frac{36}{120} = \frac{3}{10} = 0.3\,. \qquad \blacksquare$$

SOLUTION TO 2. If A is the event that the first two pets to go out are black and B is the event that the last two include a black one and white one, then we are being asked to compute the conditional probability of A given B, $P(A|B)$. We will do so using the formula $P(A|B) = \dfrac{P(AB)}{P(B)}$.

There are $\binom{3}{1} = 3$ and $\binom{2}{1} = 2$ ways to pick one of three black pets and one of two white pets, respectively, to go out last and $2! = 2$ ways to arrange the two chosen pets. Further, there are $3! = 6$ ways to arrange the other three pets. It follows that $\binom{3}{1} \cdot \binom{2}{1} \cdot 2! \cdot 3! = 3 \cdot 2 \cdot 2 \cdot 6 = 72$ oucomes in B; since there are a total of $5! = 120$ possible outcomes, all equally likely, it follows that $P(B) = \frac{72}{120} = \frac{3}{5} = 0.6$.

As noted above, there are $\binom{3}{1} = 3$ and $\binom{2}{1} = 2$ ways to pick one of three black pets and one of two white pets, respectively, to go out last and $2! = 2$ ways to arrange the two chosen pets. The other two black pets must go first, then, and there are $2! = 2$ ways to arrange which goes out first and which second. Note that there is only possibility, the remaining white pet, to go out third. It follows that there are $\binom{3}{1} \cdot \binom{2}{1} \cdot 2! \cdot 2! \cdot 1 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 24$ outcomes in AB; since there are a total of $5! = 120$ possible outcomes, all equally likely, it follows that $P(AB) = \frac{24}{120} = \frac{1}{5} = 0.2$.

Thus

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3} = 0.3333\ldots \qquad \blacksquare$$

Quiz #4. Wednesday, 10 July, 2013. *[15 minutes]*

A fair tetrahedral (four-sided) die has faces numbered 1 through 4. The random variable X is the sum of the numbers obtained by rolling this die twice.

1. Determine the probability mass function of X. *[4]*
2. Use the probability mass function to help find $P\left(4 \leq X \leq 6\right)$. *[2]*

SOLUTION TO 1. The sample space consists of all pairs (a, b) with $1 \leq a,\ b \leq 4$, so the possible sums include all the integers from $2 = 1 + 1$ to $8 = 4 + 4$. Note that there are 16 equally likely outcomes in the sample space. We will determine what the probability mass function $p(x)$ of X is using the definition:

- If $x \neq 2$, 3, 4, 5, 6, 7, or 8, then $p(x) = 0$.
- $p(2) = P(X = 2) = P\left(\{\,(1,1)\,\}\right) = \frac{1}{16}$.
- $p(3) = P(X = 3) = P\left(\{\,(2,1),\ (1,2)\,\}\right) = \frac{2}{16} = \frac{1}{8}$.
- $p(4) = P(X = 4) = P\left(\{\,(3,1),\ (2,2),\ (1,3)\,\}\right) = \frac{3}{16}$.
- $p(5) = P(X = 5) = P\left(\{\,(4,1),\ (3,2),\ (2,3),\ (1,4)\,\}\right) = \frac{4}{16} = \frac{1}{4}$.
- $p(6) = P(X = 6) = P\left(\{\,(4,2),\ (3,3),\ (2,4)\,\}\right) = \frac{3}{16}$.
- $p(7) = P(X = 7) = P\left(\{\,(4,3),\ (3,4)\,\}\right) = \frac{2}{16} = \frac{1}{8}$.
- $p(8) = P(X = 8) = P\left(\{\,(4,4)\,\}\right) = \frac{1}{16}$.

To summarize:

$$p(x) = \begin{cases} \frac{1}{16} & x = 2 \text{ or } 8 \\ \frac{1}{8} & x = 3 \text{ or } 7 \\ \frac{3}{16} & x = 4 \text{ or } 6 \\ \frac{1}{4} & x = 5 \\ 0 & \text{otherwise} \quad \blacksquare \end{cases}$$

SOLUTION TO 2. Since X must be an integer, and consulting the solution to question 1 above,

$$\begin{aligned} P\left(4 \leq X \leq 6\right) &= P(X = 4) + P(X = 5) + P(X = 6) \\ &= p(4) + p(5) + p(6) \\ &= \frac{3}{16} + \frac{1}{4} + \frac{3}{16} = \frac{10}{16} = \frac{5}{8}. \quad \blacksquare \end{aligned}$$

Quiz #5. Wednesday, 17 July, 2013. *[10 minutes]*

A fair icosahedral (*i.e.* 20-sided) die, with faces numbered 1 through 20, is rolled; the roll is a success if the number obtained is divisible by at least one of 4 or 5, and a failure otherwise. Let X be the number of successes in five rolls of the die.

1. Compute the expected value $E(X)$ of X. *[3]*
2. Compute the standard deviation of σ_X of X. *[3]*

Solution to 1. There are 20 possible outcomes to each roll (namely 1, 2, ..., and 20), of which 8 count as successes (namely 4, 5, 8, 10, 12, 15, 16, and 20). Since the die is fair, any outcome is as likely as any other, so the probability of success on any one roll is $p = \frac{8}{20} = 0.4$. Using the formula for the expected value of repeated Bernoulli trials (*i.e.* the expected value of the binomial distribution), we have $E(X) = np = 5 \cdot 0.4 = 2$. $\square$

Solution to 2. From the solution to 1 above, we have $p = 0.4$ as the probability of success on each roll, so the corresponding probability of failure is $q = 1 - p = 1 - 0.4 = 0.6$. Plugging these into the formula for the standard deviation of repeated Bernoulli trials, we have $\sigma_X = \sqrt{npq} = \sqrt{5 \cdot 0.4 \cdot 0.6} = \sqrt{1.2} \approx 1.095$. $\blacksquare$

Note: One *could* work out both $E(X)$ and σ_X from their general definitions here, but it would take much longer ...

Quiz #6. Monday, 22 July, 2013. *[15 minutes]*

Do *one* (1) of 1 or 2 below.

1. A fair die is rolled. The result is a success if the roll is 3 or 6, and is considered a failure otherwise. Let X be the number of rolls required for the first success to occur.

   a. What is the probability mass function of X? *[3]*

   b. Find the expected value $E(X)$ and standard deviation σ_X of X. *[3]*

2. Suppose the probability density function of the continuous random variable X is $f(x) = \begin{cases} \frac{1}{2}\cos(x) & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & x < -\frac{\pi}{2} \text{ or } x > \frac{\pi}{2} \end{cases}$. Compute $P(0 \leq X \leq 2)$. *[6]*

Solution to 1. *a.* The die has six equally likely outcomes, of which two count as a success. Thus $p = P(\text{success}) = \frac{1}{3}$ and $q = P(\text{failure}) = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$. The possible values of X are 1, 2, 3, ... , so the probability mass function of X is given by

$$\begin{aligned}
p(1) &= P(X = 1) = p = \tfrac{1}{3} \\
p(2) &= P(X = 2) = qp = \tfrac{2}{3} \cdot \tfrac{1}{3} = \tfrac{2}{9} \\
p(3) &= P(X = 3) = q^2 p = \left(\tfrac{2}{3}\right)^2 \tfrac{1}{3} = \tfrac{4}{27} \\
&\vdots \\
p(k) &= P(X = k) = q^{k-1} p = \left(\tfrac{2}{3}\right)^{k-1} \tfrac{1}{3} = \tfrac{2^{k-1}}{3^k} \\
&\vdots
\end{aligned}$$

and $p(x) = 0$ for $x \neq 1, 2, 3, \ldots$ □

b. The random variable X has a geometric distribution – the fact that $p(k) = P(X = k) = q^{k-1}p$ is a big clue! – means that we can simply use the appropriate formulas: $E(X) = \dfrac{1}{p} = \dfrac{1}{\frac{1}{3}} = 3$ and $\sigma_X = \dfrac{\sqrt{q}}{p} = \dfrac{\sqrt{\frac{2}{3}}}{\frac{1}{3}} = \sqrt{6}$. ■

Solution to 2. Since $-\frac{\pi}{2} < 0 < \frac{\pi}{2} < 2$, the interval $[0, 2]$ includes one point, namely $x = \frac{\pi}{2}$ where the definition of $f(x)$ changes. This means that we have to break up the integral computing $P(0 \leq X \leq 2)$ accordingly:

$$\begin{aligned}
P(0 \leq X \leq 2) &= \int_0^2 f(x)\,dx = \int_0^{\pi/2} \frac{1}{2}\cos(x)\,dx + \int_{\pi/2}^2 0\,dx = \frac{1}{2}\sin(x)\Big|_0^{\pi/2} + 0 \\
&= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0) = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 0 = \frac{1}{2} \qquad ■
\end{aligned}$$

Quiz #7. Wednesday, 23 July, 2013. *[15 minutes]*

1. Compute the expected value $E(X)$ and variance $\mathrm{Var}(X)$ of the continuous random variable X with probability density function $f(x) = \begin{cases} 3x^{-4} & 1 \leq x \\ 0 & x < 1 \end{cases}$. *[6]*

SOLUTION. Here goes:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{1} 0x \cdot 0\, dx + \int_{1}^{\infty} x \cdot 3x^{-4}\, dx = 0 + \int_{1}^{\infty} 3x^{-3}\, dx \\
&= \lim_{t \to \infty} \int_{1}^{t} 3x^{-3}\, dx = \lim_{t \to \infty} 3 \frac{x^{-2}}{-2} \Big|_{1}^{t} = \lim_{t \to \infty} \frac{-3}{2x^2} \Big|_{1}^{t} = \lim_{t \to \infty} \left[\left(\frac{-3}{2t^2} \right) - \left(\frac{-3}{2 \cdot 1^2} \right) \right] \\
&= \lim_{t \to \infty} \frac{3}{2} \left[1 - \frac{1}{t^2} \right] = \frac{3}{2}[1 - 0] = \frac{3}{2},
\end{aligned}$$

since as $t \to \infty$, $t^2 \to \infty$ too, so $\frac{1}{t^2} \to 0$.

$$\begin{aligned}
\mathrm{Var}(X) &= E\left((X - E(X))^2 \right) = E\left(X^2 \right) - (E(X))^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left(\frac{3}{2} \right)^2 \\
&= \int_{-\infty}^{1} x^2 \cdot 0\, dx + \int_{1}^{\infty} x^2 \cdot 3x^{-4}\, dx - \frac{9}{4} = 0 + \int_{1}^{\infty} 3x^{-2}\, dx - \frac{9}{4} \\
&= \left[\lim_{t \to \infty} \int_{1}^{t} 3x^{-2}\, dx \right] - \frac{9}{4} = \left[\lim_{t \to \infty} 3 \frac{x^{-1}}{-1} \Big|_{1}^{\infty} \right] - \frac{9}{4} = \left[\lim_{t \to \infty} \frac{-3}{x} \Big|_{1}^{\infty} \right] - \frac{9}{4} \\
&= \lim_{t \to \infty} \left[\frac{-3}{t} - \frac{-3}{1} \right] - \frac{9}{4} = \left[\lim_{t \to \infty} \left(3 - \frac{3}{t} \right) \right] - \frac{9}{4} = (3 - 0) - \frac{9}{4} = \frac{3}{4},
\end{aligned}$$

since as $t \to \infty$, $\frac{1}{t} \to 0$. ■

Quiz #8. Monday, 29 July, 2013. *[15 minutes]*

Recall that a continuous exponential random variable X with parameter $\lambda > 0$ has density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ and cumulative distribution function $F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

1. Find the *median* of X, *i.e.* the m such that $P(X \leq m) = P(X > m) = \frac{1}{2}$, in terms of λ. *[6]*

Solution. $P(X \leq m) = F(m)$, by the definition of cumulative distribution function, so we really need to solve the equation $F(m) = 1 - e^{-\lambda m} = \frac{1}{2}$. (Note that m must be > 0, since $F(x) = 0$ for $x \leq 0$.) Here goes:

$$\begin{aligned} 1 - e^{-\lambda m} = \frac{1}{2} \quad & \Longleftrightarrow \quad e^{-\lambda m} = 1 - \frac{1}{2} = \frac{1}{2} \\ & \Longleftrightarrow \quad -\lambda m = \ln\left(e^{-\lambda m}\right) = \ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2) = 0 - \ln(2) = -\ln(2) \\ & \Longleftrightarrow \quad m = \frac{-\ln(2)}{-\lambda} = \frac{\ln(2)}{\lambda} \qquad \blacksquare \end{aligned}$$

Quiz #9. Wednesday, 31 July, 2013. *[15 minutes]*

Do *one* (1) of 1 or 2 below.

1. If X is a Poisson random variable with parameter $\lambda > 0$, such that $p(0) = P(X = 0) = \frac{1}{2}$ and $p(1) = P(X = 1) = \frac{1}{2}\ln(2)$, what is $p(2) = P(X = 2)$? *[6]*
2. Suppose X is a normal random variable with parameters $\mu = 2$ and $\sigma = 1$. Show that $P(-1 \leq X \leq 5) \geq \frac{8}{9}$. *[6]*

SOLUTION TO 1. A Poisson random variable with parameter λ has probability mass function $p(x) = P(X = x) = \begin{cases} \frac{\lambda^x}{x!}e^{-\lambda} & x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$. Thus $\dfrac{1}{2} = p(0) = \dfrac{\lambda^0}{0!}e^{-\lambda} = \dfrac{1}{1}e^{-\lambda} = e^{-\lambda}$ and $\dfrac{1}{2}\ln(2) = p(1) = \dfrac{\lambda^1}{1!}e^{-\lambda} = \dfrac{\lambda}{1}e^{-\lambda} = \lambda e^{-\lambda}$. Hence $\lambda = \dfrac{\lambda e^{-\lambda}}{e^{-\lambda}} = \dfrac{p(1)}{p(0)} = \dfrac{\frac{1}{2}\ln(2)}{\frac{1}{2}} = \ln(2)$, and so

$$p(2) = \frac{\lambda^2}{2!}e^{-\lambda} = \frac{(\ln(2))^2}{2!}e^{-\ln(2)} = \frac{(\ln(2))^2}{2} \cdot \frac{1}{e^{\ln(2)}} = \frac{(\ln(2))^2}{2} \cdot \frac{1}{2} = \frac{(\ln(2))^2}{4} \approx 0.12011 \quad \square$$

SOLUTION TO 2. We'll use Chebyshev's Inequality, *i.e.* if X is a random variable with $\mu = E(X)$ and $\text{Var}(X) = \sigma^2$, then $P(|X - \mu| \geq t) \leq \dfrac{\sigma^2}{t^2}$ for any $t > 0$. Recall that a normal random variable X with parameters μ and σ, $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. Thus

$$\begin{aligned} P(-1 \leq X \leq 5) &= P(-3 \leq X - 2 \leq 3) = P(|X - 2| \leq 3) \\ &= 1 - P(|X - 2| \geq 3) \geq 1 - \frac{1^2}{3^2} = 1 - \frac{1}{9} = \frac{8}{9}, \end{aligned}$$

since with $\mu = 1$ and $\sigma = 1$ (as $\sigma^2 = 1$ and $\sigma > 0$) here, we have $P(|X - 2| \geq 3) \leq \dfrac{1^2}{3^2} = \dfrac{1}{9}$. ■