Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2013

FINAL EXAMINATION
Tuesday, 6 August, 2013

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do both of parts **H** and **T**, and, if you wish, part **E**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; $\leq \Gamma(11)^{\Gamma(11)}$ neurons.

Part H. Do all of **1–5**. *[Subtotal = 70/100]*

1. Five cards are drawn at random, one at a time and without replacement, from a standard 52-card deck.

a. What are the sample space and the probability of each outcome? *[4]*

b. Let F be the event that all five belong to the same suite and let A be the event that exactly one of the five is an ace. Determine whether the events F and A are independent or not. *[6]*

2. A cubical box has a random number between 2 and 4 as the length of a side.

a. What is the expected value of the volume of the box? *[9]*

b. What is the probability that the volume of the box is at most 27? *[6]*

3. A fair coin is tossed, and then tossed again until it comes up the same way the way it did on the first toss or three more tosses have taken place, whichever comes first. Let X be the total number of tosses.

a. What is the probability mass function of X? *[8]*

b. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. *[7]*

4. Suppose $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x-2 & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$ is the probability density function of the continuous random variable X. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. *[15]*

5. Suppose X is a geometric random variable such that $P(X = 2) = \frac{1}{4}$.

a. Find $p = P(X = 1)$. *[8]*

b. Estimate $P(X \geq 5)$ using Markov's Inequality, and then compute it exactly. *[7]*

*[Parts **T** and **E** are on page 2.]*

Part T. Do any *two* (2) of **6–9**. *[Subtotal = 30/100]*

6. Suppose X is a continuous random variable with a probability density function $f(x)$ such that $f(x) = f(-x)$ for all x.

a. Show that $\mathrm{E}(X) = 0$. *[6]*

b. Verify that if we also have that $\mathrm{Var}(X) = \frac{1}{2}$, then $P(0 \leq X < 1) \geq \frac{3}{8}$. *[9]*

7. Suppose X is a random variable with $\mathrm{E}(X) > 0$, and suppose X_1, X_2, X_3, ... are independent random variables with a distribution identical to X. (You can think of the X_i as being independent copies of X.) Show that if $M > 0$, then $\lim_{n\to\infty} P\left(X_1 + X_2 + \cdots + X_n > M\right) = 1$. *[15]*

8. Balls are drawn randomly, one at a time, from an urn which initially contains three white and three black balls. If the ball drawn is white, it is put back in the urn; if the ball drawn is black, it is not put back in the urn. Let X be the number of draws taken until a white ball comes up for a second time.

a. Find the probability mass function of X. *[7]*

b. Compute $\mathrm{E}(X)$ and σ_X. *[4]*

c. Let A be the event that the next to last ball drawn is black and let B be the event that $X = 4$. Determine whether A and B are independent or not. *[4]*

9. If every pair among the events A, B, and C are independent of each other, does it follow that $P(ABC) = P(A)P(B)P(C)$? Show that it does or give an example to show that it doesn't. *[15]*

[Total = 100]

Part E. Bonus!

:). In series of games numbered 1, 2, 3, ... , the winning number in the nth game is randomly chosen from the set $\{1, 2, \ldots, n+2\}$. Kosh bets on 1 in each game and intends to keep playing until winning once. What is the probability that Kosh will have to play forever? *[2]*

(:. Write an original little poem about probability or mathematics in general. *[2]*

ENJOY THE REST OF THE SUMMER!