Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Summer 2013

SOLUTIONS TO THE FINAL EXAMINATION
Tuesday, 6 August, 2013

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do both of parts **H** and **T**, and, if you wish, part **E**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; $\leq \Gamma(11)^{\Gamma(11)}$ neurons.

Please keep in mind that most of these problems can be solved in different ways, so your solutions may be correct even if they doesn't look like the ones given below. Not to mention the probability of errors in the solutions given below. (Quick! *Compute the expected value of the number of errors in the solutions, assuming that ...)*

Part H. Do all of **1**–**5**. *[Subtotal = 70/100]*

1. Five cards are drawn at random, one at a time and without replacement, from a standard 52-card deck.

a. What are the sample space and the probability of each outcome? *[4]*

b. Let F be the event that all five belong to the same suite and let A be the event that exactly one of the five is an ace. Determine whether the events F and A are independent or not. *[6]*

SOLUTION. **a.** The sample space consists of all possible sequences of five distinct cards from the deck of 52. There are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!} = \frac{52!}{(52-5)!} = \binom{52}{5} \cdot 5! = 311,875,200$ outcomes. Each has a probability of $\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50} \cdot \frac{1}{49} \cdot \frac{1}{49} = \frac{1}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{1}{311,875,200}$. □

b. There are four suites and thirteen cards in each suite, so $P(F) = \frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$, and there are four aces and 48 other cards, and one of five positions the ace could occur in, so $P(A) = \frac{\binom{5}{1} \cdot 4 \cdot 48 \cdot 47 \cdot 46 \cdot 45}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$. Each of the four suites has one ace and twelve other cards, so $P(FA) = \frac{\binom{5}{1} \cdot 4 \cdot 1 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$. It follows that

$$\begin{aligned}
P(F)P(A) &= \frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \cdot \frac{\binom{5}{1} \cdot 4 \cdot 48 \cdot 47 \cdot 46 \cdot 45}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\
&= \frac{12 \cdot 11 \cdot 10 \cdot 9}{51 \cdot 50 \cdot 49 \cdot 48} \cdot \frac{\binom{5}{1} \cdot 4 \cdot 48 \cdot 47 \cdot 46 \cdot 45}{52 \cdot 51 \cdot 50 \cdot 49} \\
&= \frac{\binom{5}{1} \cdot 4 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \cdot \frac{48 \cdot 47 \cdot 46 \cdot 45}{51 \cdot 50 \cdot 49} \\
&= P(FA) \cdot \frac{48 \cdot 47 \cdot 46 \cdot 45}{51 \cdot 50 \cdot 49} \neq P(FA),
\end{aligned}$$

since $\frac{48 \cdot 47 \cdot 46 \cdot 45}{51 \cdot 50 \cdot 49} \approx 37.37431 \neq 1$. Thus A and F are not independent. ■

2. A cubical box has a random number between 2 and 4 as the length of a side.

a. What is the expected value of the volume of the box? *[9]*

b. What is the probability that the volume of the box is at most 27? *[6]*

SOLUTION. **a.** A number chosen at random from the intervabigskipl $[2, 4]$ has the uniform probability density $f(x) = \begin{cases} \frac{1}{2} & 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$, since $\frac{1}{4-2} = \frac{1}{2}$. The volume of a cubical box of side length x is $V = x^3$, so

$$\begin{aligned} \mathrm{E}(V) &= \int_{-\infty}^{\infty} x^3 f(x)\,dx = \int_{-\infty}^{2} x^3 0\,dx + \int_{2}^{4} x^3 \frac{1}{2}\,dx + \int_{4}^{\infty} x^3 0\,dx \\ &= 0 + \frac{1}{2} \cdot \left.\frac{x^4}{4}\right|_2^4 + 0 = \frac{1}{8}\left(4^4 - 2^4\right) = \frac{1}{8}(256 - 16) = \frac{240}{8} = 30\,. \qquad \square \end{aligned}$$

b. $V = x^3 \leq 27$ exactly when $x \leq 3$, so

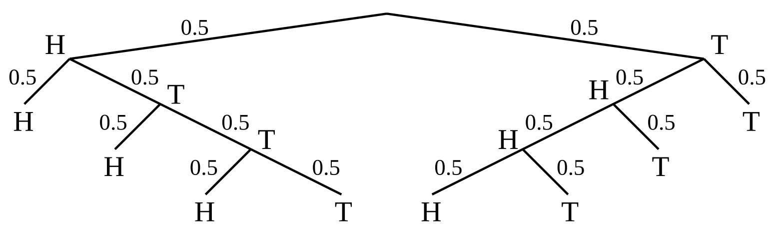
$$\begin{aligned} P(V \leq 27) = P(x \leq 3) &= \int_{-\infty}^{3} f(x)\,dx = \int_{-\infty}^{2} 0\,dx + \int_{2}^{3} \frac{1}{2}\,dx \\ &= 0 + \left.\frac{1}{2}x\right|_2^3 = \frac{1}{2}(3 - 2) = \frac{1}{2}\,. \qquad \square \end{aligned}$$

3. A fair coin is tossed, and then tossed again until it comes up the same way the way it did on the first toss or three more tosses have taken place, whichever comes first. Let X be the total number of tosses.

a. What is the probability mass function of X? *[8]*

b. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. *[7]*

SOLUTION. **a.** Here is a tree diagram for this process:



The sample space is $S = \{HH, TT, HTH, THT, HTTH, THHT, HTTT, THHH\}$, and, since the coin is fair, the probability of each outcome is $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$, where n is the number of tosses in that outcome.

Looking at the outcomes and their probabilities, and adding up appropriately, we see that $X = 2$, 3, or 4, and:

$$P(X = 2) = P(HH) + P(TT) = 2 \cdot \frac{1}{4} = \frac{1}{2}$$
$$P(X = 3) = P(HTH) + P(THT) = 2 \cdot \frac{1}{8} = \frac{1}{4}$$
$$P(X = 4) = P(HTTH) + P(THHT) + P(HTTT) + P(THHH) = 4 \cdot \frac{1}{16} = \frac{1}{4}$$

Thus $p(x) = P(X = x) = \begin{cases} \frac{1}{2} & x = 2 \\ \frac{1}{4} & x = 3 \text{ or } 4 \\ 0 & \text{otherwise.} \end{cases}$ □

b. $\mathrm{E}(X) = \sum xp(x) = 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{11}{4}$ and $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - [E(X)]^2 = \left[\sum x^2 p(x)\right] - \left[\frac{11}{4}\right]^2 = \left[2^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{4} + 4^2 \cdot \frac{1}{4}\right] - \frac{121}{16} = \frac{33}{4} - \frac{121}{16} = \frac{132}{16} - \frac{121}{16} = \frac{11}{16}$. ■

4. Suppose $f(x) = \begin{cases} x & 0 \le x \le 1 \\ x - 2 & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$ is the probability density function of the continuous random variable X. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. *[15]*

Solution. Here goes:

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\
&= \int_{-\infty}^{0} x0\, dx + \int_{0}^{1} x \cdot x\, dx + \int_{1}^{2} x0\, dx + \int_{2}^{3} x(x-2)\, dx + \int_{3}^{\infty} x0\, dx \\
&= 0 + \left.\frac{x^3}{3}\right|_0^1 + 0 + \left.\left(\frac{x^3}{3} - x^2\right)\right|_2^3 + 0 = \frac{1}{3} - 0 + \left(\frac{27}{3} - 9\right) - \left(\frac{8}{3} - 4\right) \\
&= \frac{1}{3} + 0 - \left(-\frac{4}{3}\right) = \frac{5}{3} \approx 1.67
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left[\frac{5}{3}\right]^2 \\
&= \int_{-\infty}^{0} x^2 0\, dx + \int_{0}^{1} x^2 \cdot x\, dx + \int_{1}^{2} x^2 0\, dx + \int_{2}^{3} x^2(x-2)\, dx + \int_{3}^{\infty} x^2 0\, dx - \frac{25}{9} \\
&= 0 + \left.\frac{x^4}{4}\right|_0^1 + 0 + \left.\left(\frac{x^4}{4} - \frac{2x^3}{3}\right)\right|_2^3 + 0 - \frac{25}{9} \\
&= \frac{1}{4} - 0 + \left(\frac{81}{4} - \frac{54}{3}\right) - \left(\frac{16}{4} - \frac{16}{3}\right) - \frac{25}{9} = \frac{66}{4} - \frac{38}{3} - \frac{25}{9} \\
&= \frac{33}{2} - \frac{38}{3} - \frac{25}{9} = \frac{297}{18} - \frac{228}{18} - \frac{50}{18} = \frac{19}{18} \approx 1.056 \qquad \blacksquare
\end{aligned}$$

5. Suppose X is a geometric random variable such that $P(X=2)=\frac{1}{4}$.

a. Find $p = P(X=1)$. *[8]*

b. Estimate $P(X \geq 5)$ using Markov's Inequality, and then compute it exactly. *[7]*

SOLUTION. **a.** A geometric random variable with parameter $0 < p < 1$ has probability mass function $p(k) = P(X = k) = (1-p)^{k-1}p$ for $k = 1, 2, 3, \ldots$ (and $p(x) = 0$ otherwise, of course). Thus $p(1) = P(X = 1) = p$ and $p(2) = P(X = 2) = (1-p)p$. Since $P(X = 2) = \frac{1}{4}$, we have $(1-p)p = p - p^2 = \frac{1}{4}$, *i.e.* $p^2 - p + \frac{1}{4} = 0$. We solve for p using the quadratic formula:

$$P(X=1) = p = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot \frac{1}{4}}}{2 \cdot 1} = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1 \pm 0}{2} = \frac{1}{2} \qquad \square$$

b. Recall that Markov's Inequality states that for a random variable $X \geq 0$ and any $t > 0$, we have$P(X \geq t) \leq \frac{\mathrm{E}(X)}{t}$. For a geometric random variable with parameter $p = \frac{1}{2}$, we do have $X > 0$, and $\mathrm{E}(X) = \frac{1}{p} = \frac{1}{1/2} = 2$, so $P(X \geq 5) \leq \frac{2}{5} = 0.4$ by Markov's Inequality.

It remains to compute $P(X \geq 5)$ exactly:

$$\begin{aligned} P(X \geq 5) &= \sum_{k=5}^{\infty} P(X=k) = \sum_{k=5}^{\infty} (1-p)^{k-1}p = \sum_{k=5}^{\infty} \left(1 - \frac{1}{2}\right)^{k-1} \frac{1}{2} = \sum_{k=5}^{\infty} \left(\frac{1}{2}\right)^{k-1} \frac{1}{2} \\ &= \sum_{k=5}^{\infty} \left(\frac{1}{2}\right)^{k} = \left(\frac{1}{2}\right)^{5} + \left(\frac{1}{2}\right)^{6} + \left(\frac{1}{2}\right)^{7} + \cdots, \end{aligned}$$

which is a geometric series with first term $a = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$ and common ratio $r = \frac{1}{2}$. Since $|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$, this series converges to $\frac{a}{1-r} = \frac{\frac{1}{32}}{1-\frac{1}{2}} = \frac{\frac{1}{32}}{\frac{1}{2}} = \frac{2}{32} = \frac{1}{16}$. Thus $P(X \geq 5) = \frac{1}{16}$. Note that $\frac{1}{16}$ is indeed $\leq \frac{2}{5}$. ∎

Part T. Do any *two* (2) of **6–9**. *[Subtotal = 30/100]*

6. Suppose X is a continuous random variable with a probability density function $f(x)$ such that $f(x) = f(-x)$ for all x.

a. Show that $\mathrm{E}(X) = 0$. *[6]*

b. Verify that if we also have that $\mathrm{Var}(X) = \frac{1}{2}$, then $P(0 \leq X < 1) \geq \frac{3}{8}$. *[9]*

SOLUTION. **a.** We need to show that $\mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)\,dx = 0$. Since $\int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} xf(x)\,dx + \int_{0}^{\infty} xf(x)\,dx$, it is sufficient to check that $\int_{-\infty}^{0} xf(x)\,dx = -\int_{0}^{\infty} xf(x)\,dx$.

$$\begin{aligned}
\int_{-\infty}^{0} xf(x)\,dx &= \lim_{t\to\infty} \int_{-t}^{0} xf(x)\,dx \qquad \text{Substitute } u=-x \text{, so } x=-u \text{ and thus } dx=(-1)\,du \text{, and } \begin{smallmatrix} x & -t & 0 \\ u & t & 0 \end{smallmatrix}. \\
&= \lim_{t\to\infty} \int_{t}^{0} (-u)f(-u)\,(-1)\,du = \lim_{t\to\infty} \int_{t}^{0} uf(u)\,du \quad (\text{As } f(-u)=f(u).) \\
&= \lim_{t\to\infty} -\int_{0}^{t} uf(u)\,du = -\lim_{t\to\infty} \int_{0}^{t} uf(u)\,du = -\int_{0}^{\infty} uf(u)\,du \\
&= -\int_{0}^{\infty} xf(x)\,dx\,, \quad \text{as required. } \square
\end{aligned}$$

b. First, note that since $f(-x) = f(x)$, an argument similar to the one in the solution to **a** above shows that $P(0 \leq X < 1) = \int_0^1 f(x)\,dx = \int_{-1}^{0} f(x)\,dx = P(-1 < X \leq 0)$. It follows that $P(0 \leq X < 1) = \frac{1}{2}P(-1 < X < 1)$. In turn, $P(-1 < X < 1) = 1 - P(|X| \geq 1)$, and we can estimate $P(|X| \geq 1)$ using Chebyshev's Inequality, using the fact that $\mathrm{Var}(X) = \sigma^2 = \frac{1}{2}$:

$$P(|X| \geq 1) \leq \frac{\sigma^2}{1^2} \leq \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

It follows that $P(-1 < X < 1) = 1 - P(|X| \geq 1) \geq 1 - \frac{1}{2} = \frac{1}{2}$, and so $P(0 \leq X < 1) = \frac{1}{2}P(-1 < X < 1) \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Unfortunately, $\frac{1}{4} \leq \frac{3}{8}$. My bad: I intended to have $\sigma = \frac{1}{2}$, not $\mathrm{Var}(X) = \sigma^2 = \frac{1}{2}$, but that's not what I put in the question … ■

7. Suppose X is a random variable with $\mathrm{E}(X) > 0$, and suppose $X_1, X_2, X_3, \ldots$ are independent random variables with a distribution identical to X. (You can think of the X_i as being independent copies of X.) Show that if $M > 0$, then $\lim_{n\to\infty} P\left(X_1 + X_2 + \cdots + X_n > M\right) = 1$. *[15]*

Solution. By the Weak Law of Large Numbers, we know that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right) = 0\,,$$

where $\mu = \mathrm{E}(X)$. Since any probability is non-negative, and

$$\begin{aligned} P\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right) &= P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu > \varepsilon\right) \\ &\quad + P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu < -\varepsilon\right)\,, \end{aligned}$$

it follows that for any $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu < -\varepsilon\right) \\ &= \lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} < \mu - \varepsilon\right) = 0\,. \end{aligned}$$

In turn, this means that for any $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} \geq \mu - \varepsilon\right) \\ &= \lim_{n\to\infty} \left[1 - P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} < \mu - \varepsilon\right)\right] = 1 - 0 = 1\,. \end{aligned}$$

Recall that we are given that $\mu = \mathrm{E}(X) > 0$. Choose $\varepsilon = \frac{\mu}{2}$, so $\mu - \varepsilon = \frac{\mu}{2} > 0$. Since $M > 0$ is fixed, there is some N such that for all $n \geq N$, $\frac{\mu}{2} > \frac{M}{n}$. Then, for every $n \geq N$, we have

$$\begin{aligned} 1 \geq P\left(X_1 + X_2 + \cdots + X_n > M\right) &= P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} > \frac{M}{n}\right) \\ &\geq P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} \geq \frac{\mu}{2}\right) \\ &= P\left(\frac{X_1 + X_2 + \cdots + X_n}{n} \geq \mu - \varepsilon\right)\,. \end{aligned}$$

Since $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} P\left(\frac{X_1+X_2+\cdots+X_n}{n} \geq \mu - \varepsilon\right) = 1$, it now follows by the Squeeze Theorem that $\lim_{n\to\infty} P\left(X_1 + X_2 + \cdots + X_n > M\right) = 1$, as desired. ■

Note: The solution above is guilding the lily a bit – I would have accepted a rather more informal argument for full credit ...

8. Balls are drawn randomly, one at a time, from an urn which initially contains three white and three black balls. If the ball drawn is white, it is put back in the urn; if the ball drawn is black, it is not put back in the urn. Let X be the number of draws taken until a white ball comes up for a second time.

a. Find the probability mass function of X. *[7]*

b. Compute $\mathrm{E}(X)$ and σ_X. *[4]*

c. Let A be the event that the next to last ball drawn is black and let B be the event that $X = 4$. Determine whether A and B are independent or not. *[4]*

Solution. **a.** The fact that black balls are not replaced means that $X = 2$, 3, 4, or 5.

$$\begin{aligned}
p(2) &= P(X=2) = P(WW) = \frac{3}{6}\cdot\frac{3}{6} = \frac{100}{400} \\
p(3) &= P(X=3) = P(BWW) + P(WBW) = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{3}{5} + \frac{3}{6}\cdot\frac{3}{6}\cdot\frac{3}{5} = \frac{9}{50} + \frac{3}{20} = \frac{132}{400} \\
p(4) &= P(X=4) = P(BBWW) + P(BWBW) + P(WBBW) \\
&= \frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}\cdot\frac{3}{4} + \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{3}{4} + \frac{3}{6}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{3}{4} = \frac{111}{400} \\
p(4) &= P(X=5) = P(BBBWW) + P(BBWBW) + P(BWBBW) + P(WBBBW) \\
&= \frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{3}{3}\cdot\frac{3}{3} + \frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{3} + \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{3}{3} + \frac{3}{6}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{3}{3} \\
&= \frac{57}{400}
\end{aligned}$$

... and, of course, $p(x) = 0$ if $X \neq 2$, 3, 4, or 5. □

b. $$\mathrm{E}(X) = \sum_{x=2}^{5} xp(x) = 2\cdot\frac{100}{400} + 3\cdot\frac{132}{400} + 4\cdot\frac{111}{400} + 5\cdot\frac{57}{400} = \frac{1325}{400} = \frac{53}{16} = 3.3125$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}\left(x^2\right) - [\mathrm{E}(X)]^2 = \left[\sum_{x=2}^{5} x^2 p(x)\right] - \left[\frac{53}{16}\right]^2 \\
&= \left[2^2\cdot\frac{100}{400} + 3^2\cdot\frac{132}{400} + 4^2\cdot\frac{111}{400} + 5^2\cdot\frac{57}{400}\right] - \frac{2809}{256} \\
&= \frac{4789}{400} - \frac{2809}{256} \approx 0.99984. \qquad \square
\end{aligned}$$

c. Note that $A = \{WBW, BWBW, WBBW, BBWBW, BWBBW, WBBBW\}$, $B =$

$\{ BBWW, BWBW, WBBW \}$, and $AB = \{ BWBW, WBBW \}$. It follows that

$$
\begin{aligned}
P(A) &= P(WBW) + P(BWBW) + P(WBBW) + P(BBWBW) + P(BWBBW) \\
&\qquad + P(WBBBW) \\
&= \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{5} + \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{3} + \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{3}{3} \\
&\qquad + \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{3}{3} = \frac{202}{400} = \frac{101}{200}, \\
P(B) &= P(BBWW) + P(BWBW) + P(WBBW) \\
&= \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{3}{4} = \frac{111}{400}, \text{ and} \\
P(AB) &= P(BWBW) + P(WBBW) = \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} + \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{3}{4} = \frac{81}{400},
\end{aligned}
$$

and thus $P(A)P(B) = \frac{101}{200} \cdot \frac{111}{400} = 0.1401375 \neq 0.2025 = \frac{81}{400} = P(AB)$, so A and B are not independent. ■

9. If every pair among the events A, B, and C are independent of each other, does it follow that $P(ABC) = P(A)P(B)P(C)$? Show that it does or give an example to show that it doesn't. *[15]*

Solution. It does not necessarily follow that $P(ABC) = P(A)P(B)P(C)$. For example, suppose we toss a coin twice times and let A be the event of getting H on the first toss, B be the event of getting H on the second toss, and C be the event of getting exactly one head on the two tosses. It is easy to check that $P(A) = P(B) = P(C) = \frac{1}{2}$, $P(AB) = P(HH) = \frac{1}{4} = P(A)P(B)$, $P(AC) = P(HT) = \frac{1}{4} = P(A)P(C)$, and $P(BC) = P(TH) = \frac{1}{4} = P(B)P(C)$. However, $P(ABC) = P(\emptyset) = 0 \neq \frac{1}{8} = P(A)P(B)P(C)$. ■

[Total = 100]

Part E. Bonus!

:). In series of games numbered 1, 2, 3, ..., the winning number in the nth game is randomly chosen from the set $\{ 1, 2, \ldots, n+2 \}$. Kosh bets on 1 in each game and intends to keep playing until winning once. What is the probability that Kosh will have to play forever? *[2]*

Solution. The probability is 0 – you figure out why! ■

(:. Write an original little poem about probability or mathematics in general. *[2]*

Solution. On your own on this one! ■

Enjoy the rest of the summer!