Trent University
Summer 2017

MATH 1350H Test

Monday, 29 May

Time: 60 minutes

Name: Solutions

Student Number: 0123456

Question	Mark	
1	____	
2	____	
3	____	
4	____	
Total	____	/40

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do **a** and any *two* (2) of **b**–**d**. *[10 = 2 + 2 × 4 each]*

Consider the lines given by the equations $x - 2y = 2$ and $2x + y = -2$.

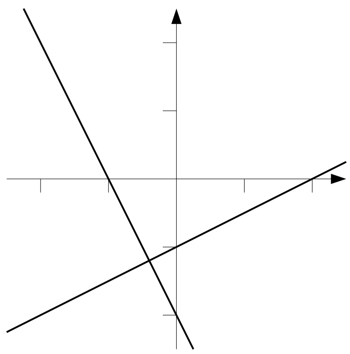
a. Sketch a graph of the two lines including their x- and y-intercepts.

b. Find the point where the lines intersect.

c. Find vector-parametric equations for each of the lines.

d. Find the angle between the lines.

SOLUTIONS. **a.** Here is a crude sketch of the two lines:



Observe that the intercepts of the line $x - 2y = 2$ are $x = 2$ on the x-axis and $y = -1$ on the y-axis, while the intercepts of the line $2x + y = -2$ are $x = -1$ on the x-axis and $y = -2$ on the y-axis. □

b. Solving the equation of the first line for x gives $x = 2y + 2$. Substituting this into the equation of the second line gives $2(2y + 2) + y = -2$, so $5y + 4 = -2$, hence $5y = -6$, and thus $y = -\frac{6}{5}$. Substituting back gives $x = 2\left(-\frac{6}{5}\right) + 2 = \frac{-12+10}{5} = -\frac{2}{5}$. Thus the lines intersect at the point $\left(-\frac{2}{5}, -\frac{6}{5}\right)$. (As a quick sanity check, this is consistent with the crude sketch drawn for **a**.) □

c. We'll use the vectors from the origin to the x-intercepts of each line, namely $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ for the base vectors, and the vectors from the x-interceopt to the y-intercepts, namely $\begin{bmatrix} 0-2 \\ -1-0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0-(-1) \\ -2-0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, for the direction vectors. Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

are vector-parametric equations for the two lines $x - 2y = 2$ and $2x + y = -2$, respectively. (s and t are the respective parameters – it's probably not a good idea to use the same parameter for both lines in most applications.) □

d. The angle between the lines is the angle between the direction vectors. $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-2)(-1) + 1(-2) = 2 - 2 = 0$, so the two direction vectors are orthogonal to each other, so the angle between the lines is a right angle (*i.e.* $90°$ or $\frac{\pi}{2}$ rad). ■

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

Let $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

a. Find the components of $\mathbf{w}$ that are, respectively, parallel to and perpendicular to $\mathbf{v}$.

b. Solve $2\mathbf{u} + 4\mathbf{w} + 5\mathbf{x} = -3\mathbf{v}$ for $\mathbf{x}$.

c. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

SOLUTIONS. **a.** The component of $\mathbf{w}$ parallel to $\mathbf{v}$ is the projection of $\mathbf{w}$ onto $\mathbf{v}$:

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

The component of $\mathbf{w}$ perpendicular to $\mathbf{v}$ is what is left when you take the component parallel to $\mathbf{v}$ away from $\mathbf{w}$:

$$\mathbf{w} - \mathrm{proj}_{\mathbf{v}}(\mathbf{w}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} \qquad \square$$

b. Just a bit of rearranging and arithmetic:

$$\begin{aligned} & 2\mathbf{u} + 4\mathbf{w} + 5\mathbf{x} = -3\mathbf{v} \\ \Longrightarrow\ & 5\mathbf{x} = -2\mathbf{u} - 3\mathbf{v} - 4\mathbf{w} \\ \Longrightarrow\ & \mathbf{x} = -\frac{2}{5}\mathbf{u} - \frac{3}{5}\mathbf{v} - \frac{4}{5}\mathbf{w} = -\frac{2}{5} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{5} \\ -\frac{6}{5} \\ -1 \end{bmatrix} \qquad \square \end{aligned}$$

c. Recall that if θ is the (smallest) angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1}{\sqrt{0^2 + 1^2 + 1^2}\sqrt{1^2 + 0^2 + 1^2}} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2},$$

so $\theta = \arccos\left(\frac{1}{2}\right) = 60° = \frac{\pi}{3}$ rad. ■

3. Consider the following system of linear equations:
$$\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 1 \\ x & + & y & & & = & 1 \\ x & + & 2y & - & z & = & 2 \end{array}$$

a. Find all the solutions, if any, of this system. *[8]*

b. Use your solution to **a** to help determine whether the vectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent. *[2]*

Solutions. **a.** We set up the augmented matrix corresponding to the given system and apply the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 2 \end{array}\right] \underset{\Longrightarrow}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \end{array}\right] \underset{\substack{R_2 - 2R_1 \\ R_3 - R_1}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\underset{(-1)R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \underset{R_3 - R_2}{\overset{R_1 - R_2}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \textit{i.e.} \quad \begin{array}{c} x + z = 0 \\ y - z = 1 \end{array}$$

Having fully reduced the matrix, we see that there are infinitely many solutions; if we set $z = t$ for a parameter t, then $x = -z = -t$ and $y = 1 + z = 1 + t$, which forms a line in $\mathbb{R}^3$. In vector-parametric form the solutions are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. $\square$

b. The three vectors are not linearly independent. If one were to replace the right-hand side column in the original augmented matrix above by all zeros and then apply the Gauss-Jordan algorithm just as above, the final augmented matrix woould be $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, representing the system $x + z = 0$ and $y - z = 0$. (Recall that the allowed row operations do not change a column of zeros.) This also has infinitely many solutions, so there must x, y, and z which are not all 0 and such that $x \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This means, by definition, that these vectors are not linearly independent. ■

4. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, and let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

a. Determine whether $\mathbf{x} \in \mathrm{Span}(S)$. *[8]*

b. Determine whether S is a linearly independent set of vectors. *[2]*

SOLUTIONS. **a.** By definition, $\mathbf{x} \in \mathrm{Span}(S)$ if there are scalars a, b, and c such that $a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *i.e.* such that $a+b+c=1$, $a+c=0$, $a+b=0$, and $a+b+c=1$. We set up the augmented matrix for this system of equations and Gauss-Jordan away:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ (-1)R_2 \\ (-1)R_3 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \textit{i.e.} \quad \begin{array}{c} a = -1 \\ b = 1 \\ c = 1 \end{array}$$

Since there is a solution, $\mathbf{x} \in \mathrm{Span}(S)$. $\square$

b. Proceeding just as in the solution to **3b**, if one were to replace the right-hand side in the calculation above with zeros, one would eventually obtain $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ as the fully reduced augmented matrix, corresponding to the unique solution $a=b=c=0$. By definition, this means that S is a linearly independent set. $\blacksquare$

[Total = 40]