Mathematics 1350H – Linear Algebra I: Matrix Algebra

Trent University, Summer 2017

Solutions to the Quizzes

Quiz #1. Wednesday, 10 May, 2017. *[10 minutes]*

Let $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$.

1. Find $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. *[2]*
2. Determine whether or not $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other. *[2]*
3. Let $c = \dfrac{1}{\|\mathbf{a}\|}$. Without actually working out the numbers, what is $\|c\mathbf{a}\|$ equal to? *[1]*

Solutions. 1. First, $\mathbf{a} + \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1) + (-1) \\ 2 + 1 \\ (-3) + 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$. Second, $\mathbf{a} - \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1) - (-1) \\ 2 - 1 \\ (-3) - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$. $\square$

2. Since the dot product $\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = (-1) \cdot (-1) + 2 \cdot 1 + (-3) \cdot 1 = 1 + 2 - 3 = 0$, $\mathbf{a}$ and $\mathbf{b}$ are indeed perpendicular to one another. $\square$

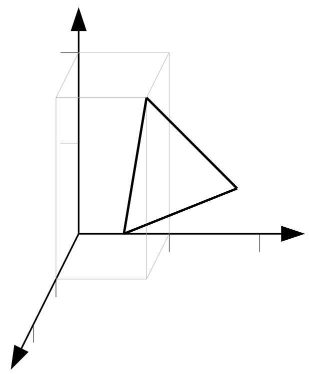
3. Note that $c = \dfrac{1}{\|\mathbf{a}\|} > 0$, because $\|\mathbf{a}\| > 0$ since $\|\mathbf{a}\|$ is the length of a vector other than $\mathbf{0}$. It follows that $\|c\mathbf{a}\| = |c| \cdot \|\mathbf{a}\| = c \cdot \|\mathbf{a}\| = \dfrac{1}{\|\mathbf{a}\|} \cdot \|\mathbf{a}\| = 1$. $\blacksquare$

Quiz #2. Monday, 15 May, 2017. *[10 minutes]*

Consider the three points $(1, 1, 2)$, $(1, 2, 1)$, and $(2, 1, 1)$ in $\mathbb{R}^3$.

1. Sketch the axes of $\mathbb{R}^3$, the three given points, and the triangle they make. *[1]*
2. Find a parametric representation of the plane passing through the given points. *[2]*
3. Find a linear equation representing the plane passing through the given points. *[2]*

SOLUTIONS. 1. Here is a fairly crude sketch:



The box guiding where $(1, 1, 2)$ should be is drawn in, just to show how it's done. Note that due to the crude perspective, the point $(2, 1, 1)$ appears to be on the y-axis. □

2. We'll go for a vector-parametric representation, using the coordinates of first point to give us the base vector,

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

and the vectors from the base point to the other two points for the direction vectors:

$$\mathbf{u} = \begin{bmatrix} 1-1 \\ 2-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2-1 \\ 1-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

This gives the following vector-parametric representation of the given plane:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x} = \mathbf{x}_0 + s\mathbf{u} + t\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \text{ for } s,\ t \in \mathbb{R}.$$

For those who don't like vector-parametric form, one could also write the parametrization coordinate by coordinate: $x = 1 + t$, $y = 1 + s$, $z = 2 - s - t$. □

3. From the solution to question 2 above, $x = 1 + t$, so $t = x - 1$, and $y = 1 + s$, so $s = y - 1$. It follows that $z = 2 - s - t = 2 - (y - 1) - (x - 1) = 4 - x - y$, and moving x and y to the left-hand side then gives us the equation $x + y + z = 4$. (You can check this answer by checking that each of the three original points satisfies this equatiion.) ■

Quiz #3. Wednesday, 17 May, 2017. *[15 minutes]*

1. Use the Gauss-Jordan method to find the point(s) of intersection, if any, of the planes in $\mathbb{R}^3$ given by the linear equations $x - y + z = 1$, $2x - y - z = 0$, and $x - 2y + 3z = 2$, respectively. *[5]*

SOLUTION. We set up the augmented matrix for the system of equations and Gauss-Jordan away:

$$
\begin{aligned}
&\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & -1 & -1 & 0 \\ 1 & -2 & 3 & 2 \end{array}\right]
\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & -1 & 2 & 1 \end{array}\right] \\
\begin{array}{c} R_1 + R_2 \\ \Longrightarrow \\ R_3 + R_2 \end{array}
&\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -1 & -1 \end{array}\right]
\begin{array}{c} \\ \Longrightarrow \\ (-1)R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \\
\begin{array}{c} R_1 + 2R_2 \\ R_2 + 3R_3 \\ \Longrightarrow \end{array}
&\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]
\end{aligned}
$$

Since the coefficient part of the final augmented matrix is a 3×3 identity matrix, we see that the three planes meet in a single point, whose coordinates we can read off the right-hand side part of the final augmented matrix. That is $x = 1$, $y = 1$, and $z = 1$ is the only solution to the given system of linear equations, so the sole common point of the three planes is $(1, 1, 1)$. ■

Quiz #4. Wednesday, 24 May, 2017. *[20 minutes]*

1. Use the Gauss-Jordan method to find all the solutions, if any, of the following system of linear equations. *[5]*

$$\begin{array}{ccccccccc} 2x & - & y & + & 5z & - & 8w & = & 6 \\ x & - & 2y & + & 10z & + & w & = & -3 \\ x & - & y & + & 7z & - & w & = & 1 \\ x & + & y & + & z & - & 5w & = & 9 \end{array}$$

Corrected Solution. As usual, we set up the corresponding augmented matrix and apply the Gauss-Jordan algorithm:

$$\left[\begin{array}{cccc|c} 2 & -1 & 5 & -8 & 6 \\ 1 & -2 & 10 & 1 & -3 \\ 1 & -1 & 7 & -1 & 1 \\ 1 & 1 & 1 & -5 & 9 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_4 \\ \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -5 & 9 \\ 1 & -2 & 10 & 1 & -3 \\ 1 & -1 & 7 & -1 & 1 \\ 2 & -1 & 5 & -8 & 6 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -5 & 9 \\ 0 & -3 & 9 & 6 & -12 \\ 0 & -2 & 6 & 4 & -8 \\ 0 & -3 & 3 & 2 & -12 \end{array}\right] \begin{array}{c} \\ -\frac{1}{3}R_2 \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -5 & 9 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & -2 & 6 & 4 & -8 \\ 0 & -3 & 3 & 2 & -12 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 + 2R_2 \\ R_4 + 3R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -3 & 5 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -4 & 0 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ R_3 \leftrightarrow R_4 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -3 & 5 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 0 & -6 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ -\frac{6}{R3} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -3 & 5 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 4R_3 \\ R_2 + 3R_3 \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{17}{3} & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \textit{[Whew!]}$$

By setting w equal to the parameter t, we can now solve for the other variables in terms of t. Thus $x = 5 + \frac{17}{3}t$, $y = 4$, $z = -\frac{2}{3}t$, and $w = t$ is a parametric description of all the solutions. The corresponding vector-parameteric form is:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{17}{3} \\ 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Note that the infinitely many solutions form a line in $\mathbb{R}^4$. ■

Quiz #5. Wednesday, 31 May, 2017. *[10 minutes]*

$$\text{Let } \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 9 \\ 2 \\ 6 \end{bmatrix}.$$

1. Compute $\mathbf{Ab}$ and $\mathbf{Ac}$. *[4]*
2. Using your work in solving question 1, compute $\mathbf{A}(2\mathbf{b} - \mathbf{c})$. *[1]*

Solutions. 1. We apply the definition of multiplying a vector by a matrix twice over:

$$\mathbf{Ab} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 - 1 \cdot 1 + 1 \cdot 4 + 0 \cdot 1 \\ 0 \cdot 3 + 1 \cdot 1 - 1 \cdot 4 + 1 \cdot 1 \\ 1 \cdot 3 + 0 \cdot 1 + 1 \cdot 4 - 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix}$$

$$\mathbf{Ac} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 - 1 \cdot 9 + 1 \cdot 2 + 0 \cdot 6 \\ 0 \cdot 5 + 1 \cdot 9 - 1 \cdot 2 + 1 \cdot 6 \\ 1 \cdot 5 + 0 \cdot 9 + 1 \cdot 2 - 1 \cdot 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 13 \\ 1 \end{bmatrix} \quad \square$$

2. We exploit the fact that multiplying vectors by a matrix respects the addition vectors and multiplication by scalars:

$$\mathbf{A}(2\mathbf{b} - \mathbf{c}) = 2\mathbf{Ab} - \mathbf{Ac} = 2 \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix} - \begin{bmatrix} -2 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -17 \\ 11 \end{bmatrix} \quad \blacksquare$$

Quiz #6. Monday, 5 June, 2017. *[12 minutes]*

1. Find the inverse matrix, if there is one, of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}$. *[5]*

Solution. We set up the "super-augmented" matrix and Gauss-Jordan the day away:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & -2 & 1 & 0 \\ 0 & 3 & 0 & -3 & 0 & 1 \end{array}\right]$$
$$\begin{array}{c} \Longrightarrow \\ \frac{1}{2}R_2 \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 0 & -3 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 0 & -\frac{3}{2} & 1 \end{array}\right]$$
$$\begin{array}{c} \\ \Longrightarrow \\ \frac{1}{3}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 + R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right]$$

Thus $\mathbf{A}$ does have an inverse matrix, namely $\mathbf{A}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{3} \\ -1 & 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{bmatrix}$.

Just to display our paranoia for all to see, we check the answer:

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{3} \\ -1 & 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 1 + 0\cdot(-1) + 1\cdot 0 & 1\cdot\frac{1}{2} + 0\cdot 0 + 1\cdot\left(-\frac{1}{2}\right) & 1\cdot\left(-\frac{1}{3}\right) + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} \\ 2\cdot 1 + 2\cdot(-1) + 0\cdot 0 & 2\cdot\frac{1}{2} + 2\cdot 0 + 0\cdot\left(-\frac{1}{2}\right) & 2\cdot\left(-\frac{1}{3}\right) + 2\cdot\frac{1}{3} + 0\cdot\frac{1}{3} \\ 3\cdot 1 + 3\cdot(-1) + 3\cdot 0 & 3\cdot\frac{1}{2} + 3\cdot 0 + 3\cdot\left(-\frac{1}{2}\right) & 3\cdot\left(-\frac{1}{3}\right) + 3\cdot\frac{1}{3} + 3\cdot\frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3
\end{aligned}$$

Since the product of the given matrix and the computed inverse is indeed the identity matrix, the computed inverse is really the inverse of the given matrix. Whew! ∎

Quiz #7. Wednesday, 7 June, 2017. *[15 minutes]*

1. Find the rank and nullity of $\mathbf{A} = \begin{bmatrix} 2 & -3 & 4 & -5 \\ -3 & 4 & -5 & 6 \\ 4 & -5 & 6 & -7 \\ -5 & 6 & -7 & 8 \end{bmatrix}$. *[5]*

Solution. First, we fully reduce $\mathbf{A}$ using the Gauss-Jordan method:

$$\begin{bmatrix} 2 & -3 & 4 & -5 \\ -3 & 4 & -5 & 6 \\ 4 & -5 & 6 & -7 \\ -5 & 6 & -7 & 8 \end{bmatrix} \begin{matrix} \Longrightarrow \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 2 & -3 & 4 & -5 \\ -3 & 4 & -5 & 6 \\ 1 & -1 & 1 & -1 \\ -5 & 6 & -7 & 8 \end{bmatrix}$$

$$\begin{matrix} R_1 \leftrightarrow R_3 \\ \\ \Longrightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -3 & 4 & -5 & 6 \\ 2 & -3 & 4 & -5 \\ -5 & 6 & -7 & 8 \end{bmatrix} \begin{matrix} \Longrightarrow \\ R_2 + 3R_1 \\ R_3 - 2R_1 \\ R_4 + 5R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & 1 & -2 & 3 \end{bmatrix}$$

$$\begin{matrix} R_1 + R_2 \\ \Longrightarrow \\ R_3 + R_2 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The fully reduced matrix has two rows that are not all 0s, so $\operatorname{rank}(\mathbf{A}) = 2$.

Second, by the Rank-Nullity Law, $\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = \#$ columns in $\mathbf{A}$. It follows that $\operatorname{nullity}(\mathbf{A}) = \#$ columns in $\mathbf{A} - \operatorname{rank}(\mathbf{A}) = 4 - 2 = 2$. ■