Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Summer 2017

Assignment #5
Due on Wednesday, 14 June.

Computing determinants using the Gauss-Jordan algorithm

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. Determinants turn up in various parts of mathematics besides linear algebra. For example, they are needed when changing coordinates when integrating in multivariate calculus.

A common problem with how determinants are usually defined - see §4.2 of the textbook - is that computing them is a lot of work unless $\mathbf{A}$ is a pretty small matrix. (It's a bit of a pain even for a 3×3 matrix with the usual definition.) Here are some facts about determinants which let you compute the determinant of a matrix using the Gauss-Jordan algorithm. For large matrices, this is usually more efficient than using the standard definition.

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If $i \neq j$ and you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, *i.e.* $\mathbf{A} \underset{R_i \leftrightarrow R_j}{\Longrightarrow} \mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, *i.e.* $\mathbf{A} \underset{cR_i}{\Longrightarrow} \mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If $i \neq j$ and you add any multiple of the jth row of $\mathbf{A}$ to the ith row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, *i.e.* $\mathbf{A} \underset{R_i + cR_j}{\Longrightarrow} \mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$.

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant, *i.e.* $|\mathbf{A}^T| = |\mathbf{A}|$.

This collection of rules could be used as the definition of the determinant of a matrix. Note that rule *v* tells us that rules *ii*–*iv* could be applied just as well to the columns of $\mathbf{A}$ as to the rows.

1. Use rules *i*–*v* to find the determinant of an $n \times n$ matrix $\mathbf{A}$ if:

a. $\mathbf{A}$ has a column or a row of zeros. *[1]*

b. $\mathbf{A}$ has two equal columns or two equal rows. *[1]*

c. $\mathbf{A}$ has rank less than n. *[1]*

2. Find the determinants of each of the two matrices below both by using the Gauss-Jordan method to fully reduce the matrices and then applying rules *i*–*v*, and by using the definition in §4.2 of the textbook (p. 265 in the 4th edition).

a. $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ *[2]* **b.** $\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 3 & 3 & 15 \end{bmatrix}$ *[5]*