Trent University

MATH 1350H Test

1 June, 2015

Time: 60 minutes

Name: Solutions

Student Number: 2718281

Question	Mark	
1	____	
2	____	
3	____	
4	____	
Total	____	/40

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

Consider the lines given by the vector-parametric equations $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $t \in \mathbb{R}$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $s \in \mathbb{R}$.

a. Find the angle between the lines.

b. Find an equation of the form $ax + by = c$ for each of the lines.

c. Find the point where the lines intersect.

SOLUTIONS. **a.** Sanity check: the given lines are not parallel since their direction vectors are not multiples of one another. Since they lines in $\mathbb{R}^2$ which are not parallel, they must intersect, so "the angle between th lines" makes sense.

The angle between the lines is the angle θ between their direction vectors. Then

$$\cos(\theta) = \frac{\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\| \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\|} = \frac{2 \cdot 3 + 2 \cdot 2}{\sqrt{2^2 + 2^2}\sqrt{3^2 + 2^2}} = \frac{10}{\sqrt{8}\sqrt{13}} \approx 0.9806$$

It follows that the angle between the lines is $\theta \approx \arccos(0.9806) \approx 0.197\ rad \approx 11.3°$. □

b. For the first line, we have $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x = 1 + 2t \\ 2 + 2t \end{bmatrix}$, *i.e.* $x = 1 + 2t$ and $y = 2 + 2t$. Rearranging the last equations, we get $\frac{x-1}{2} = t = \frac{y-2}{2}$, from which it follows that $2(x-1) = 2(y-2)$. Multiplying this out and rearranging gives us $x - y = -1$.

For the second line we have $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x = 1 + 3s \\ 2 + 2s \end{bmatrix}$, *i.e.* $x = 1 + 3s$ and $y = 2 + 2s$. Rearranging the last equations, we get $\frac{x-1}{3} = s = \frac{y-2}{2}$, from which it follows that $2(x-1) = 3(y-2)$. Rearranging this, in turn, gives us $2x - 3y = -4$. □

c. The two lines are given with the same base vector, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so they have the point $(1, 2)$ in common. (When $s = t = 0$, if it matters ...) ■

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

a. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

b. Solve the equation $\mathbf{u} - 2\mathbf{v} + 3\mathbf{w} - 4\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

c. Find the components of $\mathbf{w}$ that are, respectively, parallel to and perpendicular to $\mathbf{v}$.

SOLUTIONS. **a.** The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1 \cdot 1 + 1(-1) + 1 \cdot 1 + 1 \cdot 1}{\sqrt{1^2 + 1^2 + 1^2 + 1^2}\sqrt{1^2 + (-1)^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{4}\sqrt{4}} = \frac{2}{4} = \frac{1}{2}.$$

It follows that $\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}\ rad = 60°$. □

b. We are given that $\mathbf{u} - 2\mathbf{v} + 3\mathbf{w} - 4\mathbf{x} = \mathbf{0}$, so $4\mathbf{x} = \mathbf{u} - 2\mathbf{v} + 3\mathbf{w}$, and hence

$$\mathbf{x} = \frac{1}{4}(\mathbf{u} - 2\mathbf{v} + 3\mathbf{w}) = \frac{1}{4}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix} 1 - 2 \cdot 1 + 3 \cdot 1 \\ 1 - 2(-1) + 3(-1) \\ 1 - 2 \cdot 1 + 3(-1) \\ 1 - 2 \cdot 1 + 3 \cdot 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \\ \frac{1}{2} \end{bmatrix}. \qquad \square$$

c. The component of $\mathbf{w}$ parallel to $\mathbf{v}$ is the projection of $\mathbf{w}$ onto $\mathbf{v}$, namely

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{1 \cdot 1 + (-1)(-1) + (-1)1 + 1 \cdot 1}{1 \cdot 1 + (-1)(-1) + 1 \cdot 1 + 1 \cdot 1}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

The component of $\mathbf{w}$ perpendicular to $\mathbf{v}$ is $\mathbf{w} - \mathrm{proj}_{\mathbf{v}}(\mathbf{w}) = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$. ■

3. Consider the following system of linear equations: $\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 4 \\ x & + & 2y & + & z & = & 4 \\ x & + & y & + & 2z & = & 4 \end{array}$

a. Find all the solutions, if any, of this system. *[8]*

b. Use your answer to **a** to determine whether the vectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ are linearly dependent or independent. *[2]*

SOLUTIONS. **a.** As usual, we set up the augmented matrix and Gauss-Jordan away:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 2 & 4 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 2 & 4 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & -1 & -4 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 \leftrightarrow R_3 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & -1 & -4 \end{array}\right] \begin{array}{c} \Longrightarrow \\ -R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -1 & -4 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \Longrightarrow \\ R_3 + 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ -\frac{1}{4}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} R_1 - 3R_3 \\ R_2 + R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

It follows that there is exactly one solution to the given system of linear equations, namely $x = y = z = 1$. $\square$

b. Whether the three vectors are linearly independent or not comes down to, by definition, whether scalars a, b, and c such that $a \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ all have to be 0 or not. Setting up the appropriate augmented matrix and applying the Gauss-Jordan algorithm gives

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \end{array} \quad \cdots \quad \begin{array}{c} R_1 - 3R_3 \\ R_2 + R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

where the calculation proceeds exactly as the one above for **a**, except that the right-hand side column remains all 0s throughout. It follows that the only way to make the linear combination of the vectors add up to $\mathbf{0}$ is to have $a = b = c = 0$, so the three vectors are linearly independent. $\blacksquare$

4. As in **2**, let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. In addition, let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$.

a. Determine whether $\mathbf{x} \in \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. *[8]*

b. Determine whether $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent or independent. *[2]*

SOLUTIONS. **a.** By definition, $\mathbf{x} \in \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ if there are scalars a, b, and c such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{x}$. As always, we set up the appropriate augmented matrix and apply the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

We stop at this point because we have enough to determine that there is no solution: the last row corresponds to the equation $0a + 0b + 0c = 1$ ... It follows that $\mathbf{x}$ is not in $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. $\square$

b. By definition, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent if the only scalars a, b, and c such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ are $a = b = c = 0$. Setting up the coresponding augmented matrix and applying the first step of the Gauss-Jordan algorithm gives us:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ -\frac{1}{2}R_2 \\ -\frac{1}{2}R_3 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This means that $a = b = c = 0$ is the only solution, so $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent. $\blacksquare$

NOTE: The above solution for **b** is the brute-force approach. A more efficient way to get the job done, given the solution to **a** above, is to observe that ...

[Total = 40]