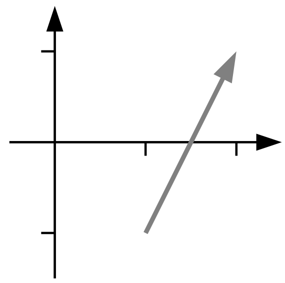
Mathematics 1350H – Linear algebra I: Matrix algebra
Trent University, Summer 2015

Quizzes

Quiz #1. Wednesday, 13 May, 2015. *[10 minutes]*

1. Find the vector in $\mathbb{R}^2$ that would take you from the point $(1,-1)$ to the point $(2,1)$ and sketch it. *[3]*
2. Find the vector in $\mathbb{R}^3$ of length 10 in the same direction as $\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$. *[2]*

Solutions. 1. The vector that would take you from $(1,-1)$ to $(2,1)$ is $\begin{bmatrix} 2-1 \\ 1-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Here is a sketch:



2. The given vector has length $\left\| \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2+0^2+4^2} = \sqrt{9+0+16} = \sqrt{25} = 5$. Scaling the vector by $\frac{10}{5} = 2$ will give a vector of length 10 in the same direction (since 2 is positive – scaling by a negative number reverses direction): $2\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 \\ 2\cdot 0 \\ 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$.
Being just a bit paranoid, we check that this really does give a vector of length 10:

$$\left\| \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix} \right\| = \sqrt{6^2+0^2+8^2} = \sqrt{36+0+64} = \sqrt{100} = 10 \qquad \blacksquare$$

Quiz #2. Wednesday, 20 May, 2015. *[12 minutes]*

Consider the lines in $\mathbb{R}^3$ given by the vector equations $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $s \in \mathbb{R}$.

1. Find the point where the lines intersect. *[0.5]*
2. Find the angle between the lines *[2]*
3. Find an equation of the form $ax + by + cz = d$ of the plane that includes both lines. *[2.5]*

SOLUTIONS. 1. The base point for both lines, $(1, 1, 1)$, is on both lines ... For the truly paranoid, observe that with $s = t = 0$, we have:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \square$$

2. The angle between the lines is the angle θ between their direction vectors. Using the formula $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}$, we get:

$$\cos(\theta) = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\|} = \frac{1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1)}{\sqrt{1^2 + 0^2 + 1^2}\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{0}{\sqrt{2}\sqrt{2}} = 0$$

It follows that $\theta = \frac{\pi}{2}$, *i.e.* the angle between the direction vectors is a right angle, so the lines are perpendicular to each other.

As a belated sanity check, note that since the two lines do intersect, by the solution to question 1, it makes sense to speak of the angle between them. $\square$

3. As an upfront sanity check, note that two intersecting lines define a plane, and the given lines do intersect by the solution to question 1.

Recall that a plane in $\mathbb{R}^3$ with normal vector $\mathbf{n}$ and base vector $\mathbf{d}$ consists of all points (x, y, z) such that $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{d}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ x \end{bmatrix}$ is the vector from the origin to the point. We can use the base vector common to both of the given lines as the base vector of the plane containing them, *i.e.* $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

It remains to find a suitable normal vector $\mathbf{n}$. This vector must be perpendicular to, *i.e.* have a dot product of zero with, both of the direction vectors of the given lines, namely $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Using the Eyeball Theorem, it's pretty easy to see that any vector with x- and z-coordinates equal to zero and a non-zero y-coordinate will do the job. For example, $\mathbf{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ will do. Thus

$$y = 0x + 1y + oz = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 = 1\,,$$

i.e. $y = 1$, is an equation of the plane.

One could, of course, obtain a suitable normal vector in other ways. For one example, one could take the cross product of the direction vectors of the lines to get a vector perpendicular to both:

$$\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k}$$

$$= (0 \cdot (-1) - 0 \cdot 1)\,\mathbf{i} - (1 \cdot (-1) - 1 \cdot 1)\,\mathbf{j} + (1 \cdot 0 - 1 \cdot 0)\,\mathbf{k} = 0\mathbf{i} + 2\mathbf{j} + 0\mathbf{k} = 2\mathbf{j} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Note that there are other ways to compute the cross product of two vectors; the method above previews determinants, which we will see more of later in the course. ■

Quiz #3. Monday, 25 May, 2015. *[20 minutes]*

1. The following system of linear equations has exactly one solution. Use the Gauss-Jordan method to find it. Show all your work. *[5]*

$$\begin{array}{ccccccc} 2x & + & y & + & 3z & = & 2 \\ x & & & + & z & = & 1 \\ x & - & y & - & z & = & 2 \end{array}$$

SOLUTION. We put the system of equations into augmented matrix form and go nuts on row operations:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 2 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & -1 & 2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ (-1)R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The final augmented matrix represents the system of equations $x = 2$, $y = 1$, and $z = -1$, which gives the solution to the original system of linear equations. ■

Quiz #4. Wednesday, 27 May, 2015. *[20 minutes]*

1. Determine whether the vectors $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are linearly dependent or independent. *[5]*

SOLUTION. As described in class and the textbook, this problem reduces to whether the only way to get scalars a, b, and c such that $a \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + c \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is to have $a = b = c = 0$ [independence] or not [dependence]. We set up the usual augmented matrix and Gauss-Jordan away:

$$\left[\begin{array}{ccc|c} -1 & 3 & 3 & 0 \\ 1 & 7 & 2 & 0 \\ 2 & 8 & 1 & 0 \end{array}\right] \begin{array}{c} (-1)R_1 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 0 \\ 1 & 7 & 2 & 0 \\ 2 & 8 & 1 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 0 \\ 0 & 10 & 5 & 0 \\ 0 & 14 & 7 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \frac{1}{10}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 14 & 7 & 0 \end{array}\right] \begin{array}{c} R_1 + 3R_2 \\ \Longrightarrow \\ R_3 - 14R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

[That is, $a - \frac{3}{2}c = 0$ and $b + \frac{1}{2}c = 0$.]

Since c can be set to any value and corresponding values of a and b found, $a = b = c = 0$ is not the only solution, so the given vectors are linearly dependent. ■

Quiz #5. Wednesday, 3 June, 2015. *[15 minutes]*

1. Find the inverse matrix of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ or show that it does not have an inverse. *[5]*

SOLUTION. We set up the "super-augmented" matrix and do the Gauss-Jordan tango:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right] \underset{R_3 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\underset{-\frac{1}{2}R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \overset{\substack{R_1 - R_2 \\ R_2 - R_3}}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

It follows that the given matrix does have an inverse, namely

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}. \qquad \blacksquare$$

Quiz #6. Monday, 8 June, 2015. *[15 minutes]*

Determine whether each of the following sets is a subspace of $\mathbb{R}^2$ or not.

1. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ 2x - y = 0 \right\}$ *[1.5]* 2. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ 2x - y = 13 \right\}$ *[1.5]*

3. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ x^2 - y = 0 \right\}$ *[2]*

SOLUTIONS. 1. U is indeed a subspace. First, if $\begin{bmatrix} x \\ y \end{bmatrix} \in U$, so $2x - y = 0$, and c is any scalar, then $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \in U$ too, because $2cx - cy = c(2x - y) = c \cdot 0 = 0$. Second, if $\begin{bmatrix} x \\ y \end{bmatrix} \in U$ and $\begin{bmatrix} s \\ t \end{bmatrix} \in U$, so $2x - y = 0$ and $2s - t = 0$, then $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x+s \\ y+t \end{bmatrix} \in U$ too, because $2(x + s) - (y + t) = (2x - y) + (2s - t) = 0 + 0 = 0$. $\square$

2. V is not a subspace because $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin V$ since $2 \cdot 0 - 0 = 0 \neq 13$. $\square$

3. W is not a subspace because it is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W$ because $1^2 - 1 = 1 - 1 = 0$, but $(-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin W$ because $(-1)^2 - (-1) = 1 + 1 = 2 \neq 0$. [It is also not closed under vector addition, by the way.] $\blacksquare$

Take-Home Quiz #7. Due on Wednesday, 10 June, 2015. *[15 minutes]*

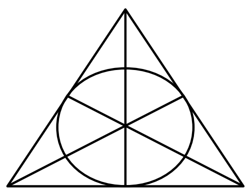
With apologies to Prof. Tolkien …

If the Númenoreans had been mathematicians, perhaps the rhyme of lore* Gandalf quotes to Pippin during the ride from Rohan to Gondor in the *The Lord of the Rings* would have been something like:

Tall ships and tall kings
Three times three,
What brought they from the foundered land
Over the flowing sea?
Seven points and seven lines
In one geometry:
Every point met three lines,
Every line met points three,
Every pair of points connected,
Every line pair intersected.

1. Draw a picture of this alternate universe Númenorean geometry. *[5]*

SOLUTION. Here's a sketch of the *Fano configuration*, also called the *Fano plane*, which is the smallest finite projective plane:

 ■

* "Tall ships and tall kings/ Three times three,/ What brought they from the foundered land/ Over the flowing sea?/ Seven stars and seven stones/ And one white tree."

Quiz #8. Wednesday, 10 June, 2015. *[15 minutes]*

1. Find a basis for the subspace $U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \begin{array}{ccccccccc} 2x & - & y & + & z & - & 2w & = & 0 \\ -x & + & 2y & + & z & + & w & = & 0 \\ x & + & y & + & 2z & - & w & = & 0 \\ 4x & + & y & + & 5z & - & 4w & = & 0 \end{array} \right\}$ of $\mathbb{R}^4$. *[5]*

SOLUTION. As usual, we use Gauss-Jordan reduction on the homogeneous system defining the subspace and use the parameteric presentation of the set of solutions to extract a basis.

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -2 & 0 \\ -1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 4 & 1 & 5 & -4 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ -1 & 2 & 1 & 1 & 0 \\ 2 & -1 & 1 & -2 & 0 \\ 4 & 1 & 5 & -4 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 & 0 \\ 0 & -3 & -3 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 & 0 \\ 0 & -3 & -3 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R2 \\ \Longrightarrow \\ R_3 + 3R_2 \\ R_4 + 3R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

That is, $x + z - w = 0$ and $y + z = 0$.

Given any values for z and w, we can clearly solve for x and y. We set $z = s$ and $w = t$ for parameters s and t. Then

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -z + w \\ -z \\ z \\ w \end{bmatrix} = \begin{bmatrix} -s + t \\ -s \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $B = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for U. ■

Quiz #9. Monday, 9 June, 2015. *[20 minutes]*

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 5 & 5 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

1. Apply the Gauss-Jordan algorithm to fully row-reduce $\mathbf{A}$. *[1]*
2. Use the results of your computation for question 1 to help find the following:
   a. The rank and nullity of $\mathbf{A}$. *[0.5]*
   b. Whether $\mathbf{A}$ is invertible or not. *[0.5]*
   c. A basis for the row space, row($\mathbf{A}$), of $\mathbf{A}$. *[1]*
   d. A basis for the column space, col($\mathbf{A}$), of $\mathbf{A}$. *[1]*
   e. A basis for the null space, null($\mathbf{A}$), of $\mathbf{A}$. *[1]*

SOLUTIONS. 1. Here goes:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 5 & 5 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ \Longrightarrow \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\begin{matrix} R_2 \leftrightarrow R_3 \\ \Longrightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \Longrightarrow \\ R_3 + R_2 \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \square$$

2a. Since the fully-reduced matrix has two non-zero rows, rank($\mathbf{A}$) $= 2$. It follows that nullity($\mathbf{A}$) $=$ (# columns of $\mathbf{A}$) $-$ rank($\mathbf{A}$) $= 4 - 2 = 2$. $\square$

2b. Since $\mathbf{A}$ is a 4×4 matrix with rank($\mathbf{A}$) $= 2 < 4$, $\mathbf{A}$ is not invertible. $\square$

2c. The non-zero rows of the reduced matrix, $\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$, are a basis for row($\mathbf{A}$). $\square$

2d. The columns of $\mathbf{A}$ corresponding to the columns of the reduced matrix in which the leading 1s of rows occur, $\begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, are a basis for col($\mathbf{A}$). $\square$

2e. By definition, the null space of $\mathbf{A}$ is the set of solutions of $\mathbf{Ax} = \mathbf{0}$, it i.e. of the homogeneous linear ystem whose coefficent matrix is $\mathbf{A}$. If we fully reduce the corresponding augmented matrix $[\mathbf{A} \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 3 & 5 & 5 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right]$, using the same operations as in the solution to 1, the final augmented matrix will be $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. This corresponds to the reduced system of equations $x - w = 0$ and $y + z + w = 0$. Setting $z = s$ and $w = t$ for parameters s and t, we get $x = t$ and $y = -s - t$, so the solutions to the homogeneous system have the vector-parametric form:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} t \\ -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

It follows that the vectors $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathrm{null}(\mathbf{A})$. ■