Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2015

FINAL EXAMINATION
Friday, 19 June, 2015

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do parts **A** and **B**, and, if you wish, part **C**. Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; ≤ 1 brain.

Part A. Do *all four* (4) of questions **1**–**4**. *[Subtotal = 64/100]*

1. Consider the following system of linear equations and its coefficient matrix **A**:

$$\begin{array}{ccccccccccc} & & v & & & + & y & & & = & 2 \\ u & + & 4v & + & 5x & + & 4y & + & z & = & 15 \\ u & & & + & x & & & + & z & = & 3 \\ & & 2v & + & 2x & + & 2y & & & = & 6 \end{array} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 4 & 5 & 4 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 2 & 0 \end{bmatrix}$$

Note that $u = v = x = y = z = 1$ is a solution of this system of linear equations.

a. Without any calculations, determine how many solutions this system has. *[3]*

b. Use Gauss-Jordan reduction to find all the solutions of this system. *[10]*

c. Find the rank and nullity of **A**. *[2]*

d. Find a basis for the column space of **A**. *[2]*

e. Find a basis for the null space of **A**. *[3]*

2. Consider the plane and line in $\mathbb{R}^3$ given by the linear equation $x+y+2z=6$ and the parametric equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $t \in \mathbb{R}$, respectively.

a. Sketch the line. *[3]*

b. Sketch the plane. *[2[*

c. Determine whether or not the line is contained in the plane. *[5]*

3. Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 8 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

a. Compute $\mathbf{A}^{-1}$, if it exists. *[10]*

b. What are the rank and nullity of **A**? Why? *[2]*

c. How many solutions $\mathbf{x}$ are there to $\mathbf{A}\mathbf{x} = \mathbf{0}$? *[2]*

d. Compute $|\mathbf{A}|$. *[5]*

4. Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^4$.

a. Find a basis for W. *[10]*

b. Find an orthogonal basis for W. *[5]*

Part B. Do *any three* (3) of questions **5–10**. *[Subtotal = 36/100]*

5. Determine whether each of the following is a subspace of $\mathbb{R}^2$ or not. *[12 = 3 × 4 each]*

a. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, xy = 0 \right\}$ **b.** $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, \cos(x+y) = 1 \right\}$

c. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, (x+y+\pi)^2 = (x-y+\pi)^2 \right\}$

6. Recall that $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{X}$ different from $\mathbf{I}_2$ which satisfies the equation $\mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = \mathbf{O}_2$, or show that $\mathbf{I}_2$ is the only matrix that satisfies the equation. *[12]*

7. Find the shortest distance from the point $(3, 3, 1)$ to the line given by the parametric equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. *[12]*

Hint: There can be no objection to removing a projection to get orthogonality!

8. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, and $T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$. Find the matrix $[T]$ of T (that is, such that $[T]\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$). *[12]*

9. Give examples in each case, or explain why none exist, of 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ which have inverses such that:

a. $\mathbf{A} - \mathbf{B}$ has no inverse. *[3]* **b.** $\mathbf{A} - \mathbf{B}$ has an inverse. *[3]*
c. $\mathbf{AB}$ has no inverse. *[3]* **d.** $\mathbf{AB}$ has an inverse. *[3]*

10. Let $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

a. Find all the eigenvalues of $\mathbf{B}$. *[5]*
b. Find a nonzero eigenvector for each eigenvalue of $\mathbf{B}$. *[5]*
c. Without computing it, what is $|\mathbf{B}|$? *[2]*

[Total = 100]

Part C. *Bonus!*

0. Write an original little poem about linear algebra or mathematics in general. *[1]*

~~FRANKENFURTER, IT'S ALL OVER!~~
GO FORTH AND ENJOY THE SUMMER!